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Matematica di base, Bigatti Robbiano   Ambrosiana

https://naturali.campusnet.unito.it/didattica/att/a08b.5472.file.pdf   Consoli, Roggero, Romagnoli, Lez. di Mat,

C & F Sbordone, Matematica per le scienze della vita

Villani, Gentili. Matematica. Comprendere e interpretare fenomeni delle scienze della vita

1^A prova in itinere (Matematica): lun 29/10/2018 9.00 aula magna

Gio 4/10 : non c'e' lezione

Si recupera gio 13/12/2018 9.00 - 11.30 aula magna (prova in itenere di fisica)

Ricevimento: MAR 8.30-9.30 Piagge 15.00-17.00 Fisica

Insiemi indicati con $A, B, C, \dots$ $X, Y \dots$

collezioni di oggetti (elementi dell'insieme) indicati

con $a, b, c \dots$ $x, y, \dots$ $A = \{ \underbrace{a, x, y}_{\text{non importa l'ordine}} \}$ $B = \{ c, a \}$

$a \in A$: "a appartiene ad A"

ϕ è l'insieme vuoto $\phi = \{ \ \}$

$\mathbb{N} = \{ 0, 1, 2, 3, \dots \}$ insieme dei numeri naturali

$n, m \in \mathbb{N}$

$\mathbb{N}_+ = \{ 1, 2, 3, \dots \} = \mathbb{N} / \{ 0 \}$

naturali positivi ↑ "privato di"

$\mathbb{Z}$ = numeri interi relativi = $\{ \dots -3, -2, -1, 0, +1, +2, \dots \}$

$n, m \in \mathbb{Z}$

$(-3) \cdot (-2) = 6$

$+ \cdot + = + \qquad + \cdot - = - \qquad - \cdot + = -$

$- \; - = +$

$\mathbb{Z}$ non è chiuso sotto divisione

$$\frac{(-2)}{(-3)} = \frac{2}{3}$$

$\mathbb{Q}$ = numeri razionali = $\left\{ \frac{m}{n} , \; m \in \mathbb{Z}, \; n \in \mathbb{Z}, \; n \neq 0 \right\}$

$$\frac{m}{n} = \frac{m \cdot p}{n \cdot p} \quad \forall\ p \in \mathbb{Z},\ p \neq 0$$

"per ogni"

$$\frac{1}{4} = \frac{1 \cdot 2}{4 \cdot 2} = \frac{2}{8} \qquad \frac{2}{3} = \frac{4}{6}$$

$$\boxed{q \in \mathbb{Q}}$$

$$q = \frac{m}{n}$$

numeratore

denominatore

Siano $q_1, q_2 \in \mathbb{Q}$

$$q_1 = \frac{m_1}{n_1} \qquad q_2 = \frac{m_2}{n_2}$$

(Es.: $q_1 = \frac{2}{7}$, $q_2 = \frac{3}{4}$)

$$q_1 \; q_2 = \frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{m_1 \cdot m_2}{n_1 \cdot n_2}$$

$$\frac{2}{7} \cdot \frac{3}{4} = \frac{\overset{3}{\cancel{6}}}{\underset{14}{\cancel{28}}} = \frac{3}{14}$$

$$q_1 + q_2 = \frac{m_1}{n_1} + \frac{m_2}{n_2} = \frac{m_1 \cdot n_2}{n_1 \cdot n_2} + \frac{m_2 \cdot n_1}{n_1 \cdot n_2} = \frac{m_1 \cdot n_2 + m_2 \cdot n_1}{n_1 \cdot n_2}$$

$$\frac{2}{7} + \frac{3}{4} = \frac{2 \cdot 4}{7 \cdot 4} + \frac{3 \cdot 7}{4 \cdot 7} = \frac{8}{28} + \frac{21}{28} =$$

$$= \frac{8+21}{28} = \frac{29}{28}$$

reciproco: se $q = \frac{m}{n} \neq 0$ $\quad \frac{1}{q} = \frac{n}{m}$

$\mathbb{Q}$ = { numeri decimali che hanno un numero finito di cifre decimali o sono periodici }

$$0,25 = \frac{0,25 \cdot 100}{1 \cdot 100} = \frac{25}{100} = \frac{1}{4}$$

numero decimale periodico: numero decimale tale che le sue cifre decimali contengono una sequenza che (da un certo punto in poi) si ripete indefinitamente

$$0,1216\underbrace{34}3434\ldots\ldots = 0,1216\overline{34}$$

$$0,\overline{563} = x \qquad 1000 \cdot x = 563,563563\ldots = 563 + \underline{\underline{0,563563\ldots}}$$

$$= 563 + x$$

$$\underline{x = 0,\underset{\uparrow}{5}63563563\ldots}$$

$$1000 \cdot x = 563 + x$$
$$-x \qquad\qquad -x$$

$$1000x - x = 563$$
$$999x = 563$$

$$x = \frac{563}{999}$$

$$3,2\overline{56} = \frac{32,\overline{56}}{10} \qquad 32,\overline{56} = 32 + 0,\overline{56}$$

$$0,\overline{56} = \frac{56}{99}$$

$$= \frac{32 + \frac{56}{99}}{10} = \frac{\overset{16}{\cancel{32}}}{\underset{5}{\cancel{10}}} + \frac{\overset{28}{\cancel{56}}}{99\cdot\underset{5}{\cancel{10}}} = \frac{16}{5} + \frac{28}{99\cdot 5}$$

$\mathbb{R}$ = insieme dei numeri reali = { tutti i numeri decimali }

$$0,\underbrace{12}\,\underbrace{121}\,\underbrace{1211}\,\underbrace{12111}\,\underbrace{121111}\ \ldots$$

Tutti i numeri reali che non sono razionali si dicono irrazionali

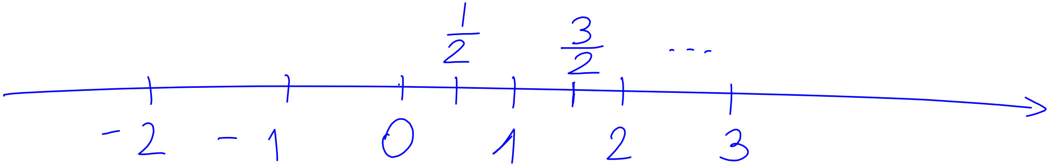
$x^2 = x \cdot x$ $\qquad$ $x^3 = x \cdot x \cdot x$

$\sqrt{x}$ è quel numero $y \mid y^2 = x$

"tale che"

$\sqrt{2}$

Retta reale



Numeri complessi

$\sqrt{-1} = ?$ $\qquad$ $i = \sqrt{-1} = y \mid y^2 = -1$ $\qquad$ $i^2 = -1$

$\mathbb{C}$ = numeri complessi = $\{ x + iy,\ x \in \mathbb{R},\ y \in \mathbb{R} \}$

parte reale

parte immaginaria

$$(x+iy)\cdot(a+ib) = xa + ixb + iya - yb$$

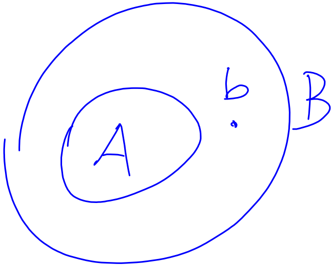
$$(a+b)(c+d) = ac + bc + ad + bd$$

Dati due insiemi A e B, si dice che A è contenuto in B, se $\forall a \in A$ è anche $a \in B$

Si scrive $A \subset B$ se $\exists\, b \in B \mid b \notin A$

$A \subseteq B$ se (può essere $A = B$

"esiste"

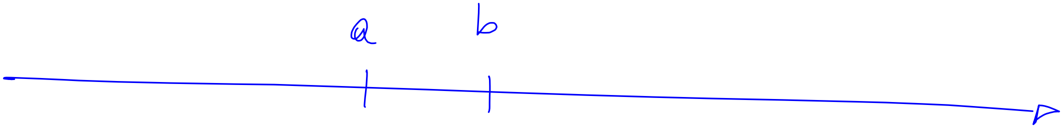


$\notin$ "non appartiene a"

$n_1 < n_2$ $n_1 \leq n_2$

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Altri esempi di insiemi $\qquad \mathbb{R}$ = retta reale

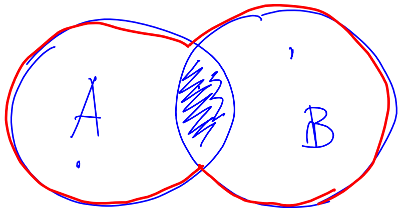


intervallo : $[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$

$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$

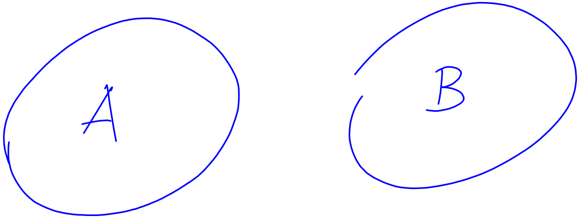
$[a, b) = \{ x \in \mathbb{R} \mid a \leq x < b \}$

Unione di insiemi $\qquad A \cup B = \{ x \mid x \in A \text{ o } x \in B \}$



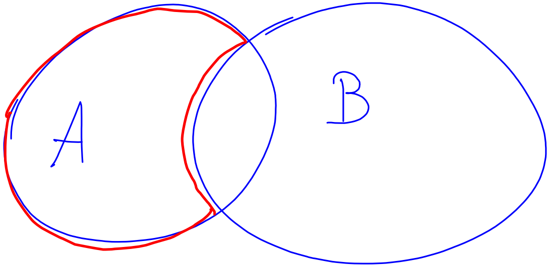
Intersezione :

$A \cap B = \{ x \mid x \in A \text{ e } x \in B \}$



$A \cap B = \emptyset$

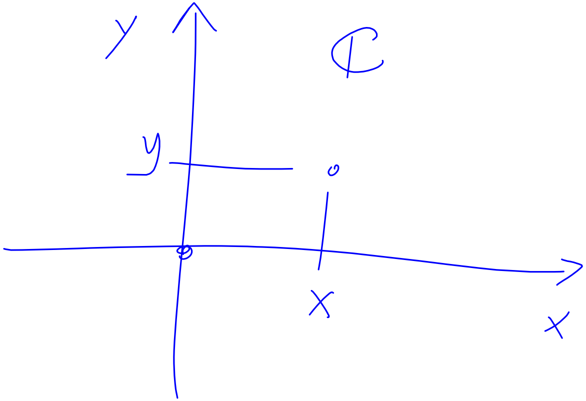
Differenza $A \setminus B = \{ x \mid x \in A, x \notin B \}$



Prodotto cartesiano insieme delle coppie di elementi

$A \times B = \{ (a, b) \mid a \in A, b \in B \}$

$\mathbb{C} = \mathbb{R} \times \mathbb{R}$

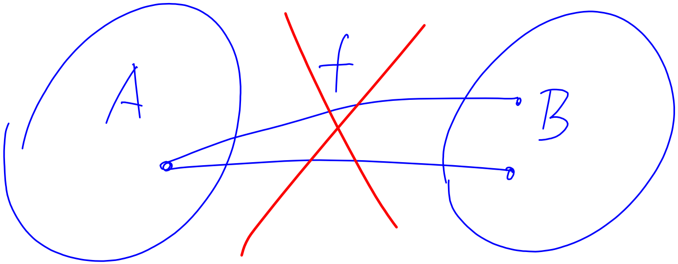


piano complesso

Funzioni tra insiemi

$f : A \longrightarrow B$

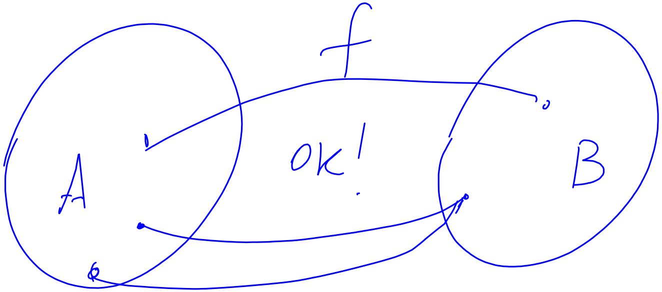
Una funzione f tra due insiemi A e B è una associazione (relazione, corrispondenza) che associa un elemento di B a ciascun elemento di A.



$f: A \longrightarrow B$

$f: a \longmapsto b$

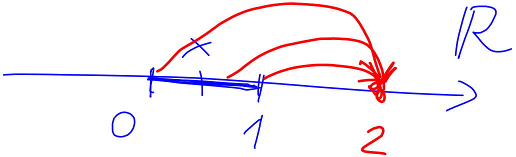
$a \in A \quad b \in B$



$$\boxed{b = f(a)}$$

A = dominio della funzione

Esempi: $A = [0, 1]$



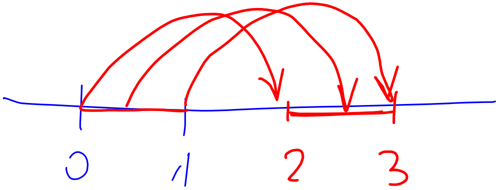
$f: A \to \mathbb{R}$ $\quad x \in A$

$f(x) = 2 \quad \forall x \in [0, 1]$

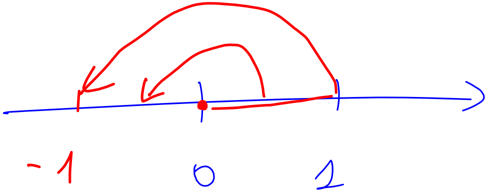
$x \in [0,1]$

$f(x) = x$ identità

$f(x) = x+2$ traslazione



$f(x) = -x$ riflessione rispetto all'origine

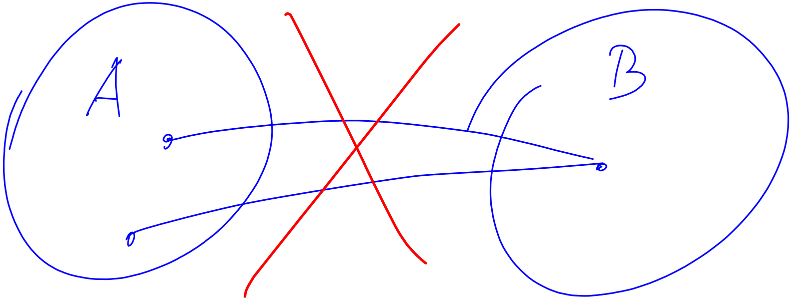


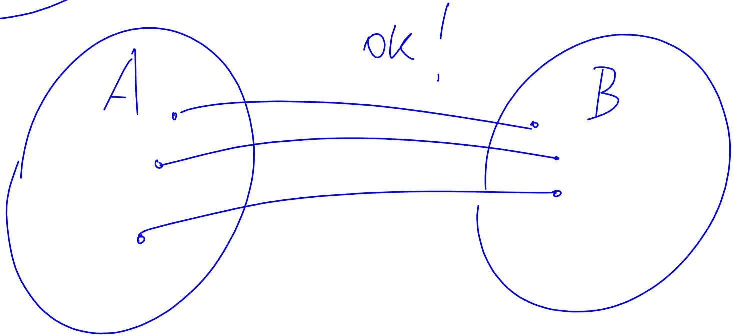
$f(x) = -x+2$

$f(x) = x^2$

$y = f(x)$ $x \in A$ $y \in B$ $f : A \to B$

Una funzione si dice <u>iniettiva</u> se elementi distinti di A finiscono in elementi distinti di B.

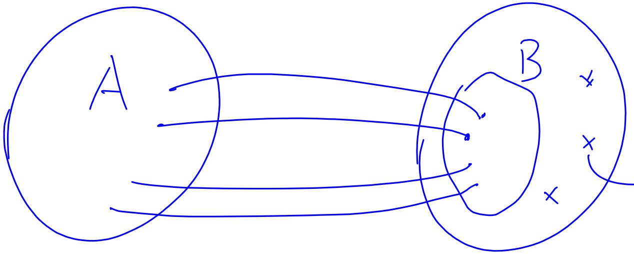




$b = f(a)$ $f : a \longmapsto b$ $f : A \to B$

b si chiama "immagine" di a

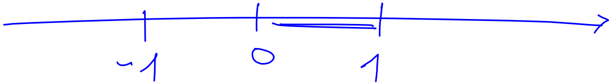
Codominio di $f = \{$ l'insieme di $b \in B \mid \exists\, a \in A \mid b = f(a) \}$ = è l'immagine di A



non sono immagine di nessun punto di A

Una funzione si dice suriettiva se B = immagine di A

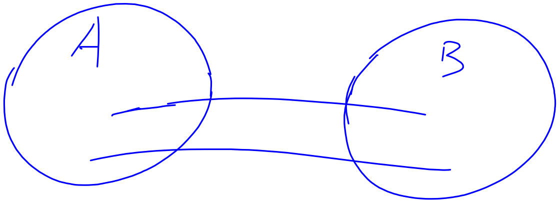
Es. $f(x) = -x$ $\qquad f : \overset{A}{[0,1]} \longrightarrow \overset{B}{[-1,0]}$



Una funzione iniettiva e suriettiva si dice biunivoca ed è invertibile

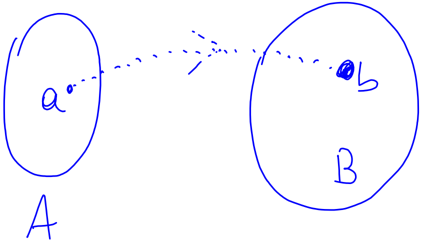
$f: A \longrightarrow B$

$a \longmapsto b$



<u>Vuol dire</u> : ad ogni elemento di A corrisponde uno e uno solo elemento di B e viceversa

Suriettività : $\forall b \in B \; \exists a \in A \mid b = f(a)$

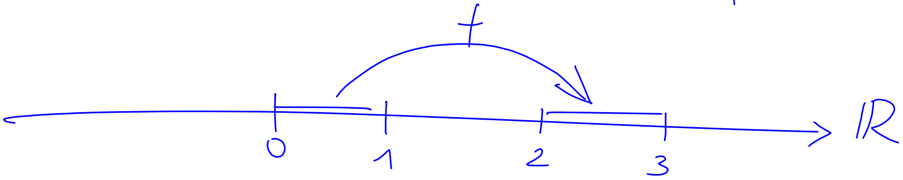


Iniettività : quell'a è unico

Quindi posso definire la funzione inversa

$f^{-1}: B \to A \qquad b \mapsto a \mid f(a) = b$

Esempio



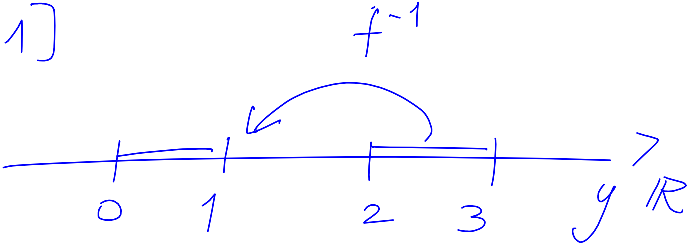
$f: [0,1] \longrightarrow [2,3]$ iniettiva e suriettiva

$f(x) = \boxed{x + 2 = y}$ ⇢ risolvo per x in funzione di y

$\exists\, f^{-1}(y)$ che mi dà x

$f^{-1}: [2,3] \longrightarrow [0,1]$

$f^{-1}(y) = y - 2$



Operazioni $A = \mathbb{R}$

Somma $+ : A \times A \longrightarrow A$

che soddisfa

proprietà associativa : $a + (b + c) =$

$$2 + (3 + 7) = (2 + 3) + 7 \qquad = (a + b) + c$$

$$2 + 10 = 5 + 7$$

proprietà commutativa : $a + b = b + a$

esistenza dell'elemento neutro (0) : $a + 0 = a \quad \forall a$

esistenza dell'opposto : $\forall a \ \exists b \ (= -a) \mid a + b = 0$

Prodotto $\cdot \quad A \times A \longrightarrow A \quad A = \mathbb{R}$

è definito dalle seguenti proprietà :

associativa : $a(bc) = (ab)c$

commutativa : $ab = ba$

$\exists$ elemento neutro (1) $\quad 1 \cdot a = a \quad \forall a \in A$

$\exists$ del reciproco (a^{-1}) : $\forall a \in A \quad a \neq 0$

$\exists\, b \; (= a^{-1}) \; / \; b \cdot a = 1$

Es: $a = 3 \quad b = \frac{1}{3}$, $a = -11 \quad b = -\frac{1}{11}$ ....

distributiva $a \cdot (b+c) = a \cdot b + a \cdot c$

$$3 \cdot (5+2) = 3 \cdot 7 = 21 = 3 \cdot 5 + 3 \cdot 2 = 15+6$$

Successioni

Una successione è una funzione

$$f : \mathbb{N} \longrightarrow \mathbb{R}$$

$0, 1, 2, \dots \qquad f(0), f(1), f(2)$

$a_0, a_1, a_2 \dots a_n$ ↖ indice

Esempio $\underline{a_n = 3n+2}$ successione aritmetica

$a_0 = 2 \quad a_1 = 5 \quad a_2 = 8 \quad \dots$

$a_n = 2^n = \underbrace{2 \cdot 2 \cdot 2 \cdot \dots 2}_{n \text{ volte}}$ successione geometrica

$a_0 = 2^0 \underset{\uparrow}{=} 1 \qquad a_1 = 2 \qquad a_2 = 2 \cdot 2 = 4 \qquad a_3 = 2 \cdot 2 \cdot 2 = 8$

per definizione ....

Successione aritmetica : a_n

tale che $a_{n+1} - a_n$ non dipende da n

$$a_{n+1} - a_n = d \quad \forall n$$ relazione di ricorrenza

$$+ a_n \qquad + a_n$$

$$a_{n+1} = a_n + d$$ a_0 è quello che è

$n=0$ $$a_1 = a_0 + d$$

$n=1$ $$a_2 = a_1 + d = a_0 + d + d = a_0 + 2d$$

$n=2$ $$a_3 = a_2 + d = a_0 + 2d + d = a_0 + 3d$$

$$\boxed{a_n = a_0 + nd}$$

Successione geometrica a_n |

$$\frac{a_{n+1}}{a_n}$$ non dipende da n

$$\frac{a_{n+1}}{a_n} = q$$ non dipende da n

$\cdot a_n$ $\cdot a_n$

$$a_{n+1} = q \cdot a_n$$

relazione di ricorrenza

a_0 è quello che è

n=0 $a_1 = q \cdot a_0$

n=1 $a_2 = q \cdot a_1 = \underbrace{q \cdot q}_{} \cdot a_0 = q^2 \cdot a_0$

$n = 2 \qquad a_3 = q \cdot a_2 = \underbrace{q \cdot q^2}_{} a_0 = q^3 \cdot a_0$

$$\ldots \qquad \underbrace{a_n = q^n a_0}_{}$$

<u>Esercizi</u> Interesse semplice, interesse composto

depositate 1000 € in banca
coll'interesse del 5% annuo

Quanto guadagno in 10 anni nei 2 casi seguenti :

1) prelevo il guadagno ogni anno

2) non lo prelevo

g_n = guadagno realizzato dopo n anni

$g_0 = 0$

il 5% di $\underline{1000\text{ €}}$ è $10\cancel{0}\cancel{0} \cdot \frac{5}{1\cancel{0}\cancel{0}}\text{ €} = 50\text{ €}$

$$\underline{g_1 = 50\text{ €}}$$

1) preleva i $\underline{\underline{50 €}}$, in banca restano 1000 €

$g_2 = 2 \cdot 50 €$ $g_3 = 3 \cdot 50 €$ ...

$\underline{g_n = n \cdot 50 €}$ $\boxed{g_{10} = 500 €}$

progressione aritmetica

2) non prelevo il guadagno

dopo 1 anno in banca avete 1000 + 50 €

nel 2° anno fruttano $\underline{\underline{\text{tutti}}}$ i 1050 €

nel 2° anno guadagno il 5% di 1050 €

$$\frac{5}{100} \cdot 1050$$

$$g_2 = 50 + \underbrace{\frac{5}{100} \cdot 1050} \quad \ldots\ldots$$

Chiamiamo $\boxed{S_n}$ la cifra depositata in banca dopo n anni

$$S_{n+1} = S_n + \text{guadagno} =$$

$$= S_n + 5\% \cdot S_n = S_n + \frac{5}{100} S_n =$$

$$= \left(1 + \frac{5}{100}\right) S_n = \left(\frac{100}{100} + \frac{5}{100}\right) S_n =$$

c

$$= \frac{\cancel{105}^{\,21}}{\cancel{100}_{\,20}} S_n = \frac{21}{20} S_n$$

$$S_{n+1} = \frac{21}{20} S_n \qquad \frac{S_{n+1}}{S_n} = \frac{21}{20}$$ non dipende da n

$$S_n = q^n S_0 \qquad q = \frac{21}{20} \qquad S_0 = 1000 \text{ €}$$

Dopo $n = 10$ anni ho

$$S_{10} = \left(\frac{21}{20}\right)^{10} 1000 \text{ €} \simeq 1629 \text{ €}$$

In 10 anni avete guadagnato $\underline{629\ €}$

Successione di Fibonacci

$a_0 = 0 \qquad a_1 = 1 \qquad a_n = a_{n-1} + a_{n-2} \quad \forall n \geq 2$

$0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad \ldots$

Permutazioni

Dato un insieme A di n oggetti distinti, si dice permutazione un qualunque loro ordinamento o allineamento

Esempio. $A = \{ \underset{\text{blu}}{B}, \underset{\text{rosso}}{R}, \underset{\text{verde}}{V} \}$

quante sono le permutazioni di 3 colori ? 6

B R V	R B V	V B R
B V R	R V B	V R B

$$6 = 3! = 3 \cdot 2 \cdot 1$$

$$n! = n \cdot (n-1) \cdots 2 \cdot 1$$

"n fattoriale" = prodotto dei primi n numeri interi

le permutazioni di n oggetti distinti sono $n!$

$$A = \{a_1, a_2, \dots, a_n\}$$

al 1° posto ne scelgo 1 su n : n possibilità

al 2° posto ne scelgo 1 su $n-1$

....

$$n \cdot (n-1) \cdot (n-2) \cdots \cdot 1 = n!$$

$$5! = 120 = 5 \cdot \underbrace{4 \cdot \underbrace{3 \cdot 2 \cdot 1}_{6}}_{24}$$

$4! = 24$

$3! = 3 \cdot 2 \cdot 1 = 6$

$6! = 720 = 6 \cdot 5!$

$a_n = n!$ successione

$a_{n+1} = (n+1)!$ $a_{n+1} = (n+1)a_n$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} = \frac{(n+1)\cdot\cancel{n}\cdot(\cancel{n-1})\cdot(\cancel{n-2})\cdots\cancel{2}\cdot 1}{\cancel{n}\cdot(\cancel{n-1})\cdot(\cancel{n-2})\cdots\cancel{2}\cdot 1} = \underline{\underline{n+1}}$$

Disposizioni
- con ripetizione
- semplici (= senza ripetizione)

Disposizioni di n elementi scelti dentro un insieme A fatto di m elementi (distinti)

$$B = \{ x_1, x_2, \dots \; x_m \}$$

disposizioni con ripetizione: ogni ordinamento (detto anche disposizione) con la possibilità di ripetere lo stesso elemento

disposizioni di n elementi di B: $\underbrace{m \cdot m \cdot m \cdot \dots m}_{n} = m^n$

disposizioni semplici di n elementi di B $(n \leq m)$

$$\underset{1^\circ}{\underline{m}} \cdot \underset{\boxed{2^\circ}}{(m-1)} \cdot \underset{\underline{3}^\circ \dots}{(m-\underline{2})} \cdots \underset{\substack{m-(n-1) \\ \underline{n}}}{(m-n+1)} = D_{m,n}$$

$$\underline{\underline{D_{m,n}}} = m\cdot(m-1)\cdot(m-2)\cdots(m-n+1) =$$

$$= \frac{m\cdot(m-1)\cdot(m-2)\cdots(m-n+1) \,\vdots\, (m-n)\cdot(m-n-1)\cdots 3\cdot 2\cdot 1}{(m-n)\cdot(m-n-1)\cdots 3\cdot 2\cdot 1} =$$

$$= \frac{m!}{(m-n)!} \quad \leftarrow | \leftarrow$$

permutazioni : sono il caso $m=n$

Verifica : $D_{n,n} \overset{?}{=} n! = n\cdot(n-1)\cdot(n-2)\cdots(1)$

$$= \frac{n!}{0!} \qquad 0! = 1$$

per definizione

Sia A un insieme di n oggetti diversi

Si dice combinazione di ordine k di n oggetti ogni sottoinsieme di A fatto di k oggetti

Il numero di tali combinazioni si indica con $\boxed{C_{n,k} = \frac{D_{n,k}}{k!}} = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$

$C_{n,k}$ si chiama anche coefficiente binomiale

Ogni disposizione semplice di k elementi è un ordinamento di quei k

Ci sono $k!$ modi di ordinare k elementi

Nelle combinazioni non mi interessa come sono ordinati i k elementi

Quindi: $\text{combinazioni} = \dfrac{\text{disposizioni}}{\text{permutazioni}}$

$$C_{n,k} = \frac{D_{n,k}}{k!}$$

$0! = 1,\ 1! = 1,\ 2! = 2$

Esempio: $A = \{B, R, V\}$ $n = 3$

Vogliamo $k = 2$ $D_{3,2} = \dfrac{3!}{(3-2)!} = 6$ $C_{3,2} = \dfrac{D_{3,2}}{2!} = 3$

Disposizioni   BR   RB   VB   : 6
BV   RV   VR

Combinazioni :   BR   BV   VR   : 3

Ricevimento   martedì dalle 8.30 alle 9.30
portineria colla sbarra
oggi a fisica dalle 16.00 alle 18.00

Matrici

"tabelle di numeri"

$$\begin{pmatrix} 1 & 3 & \sqrt{2} & \frac{1}{10} \\ 4 & 7.2 & \pi & 5/2 \end{pmatrix} \qquad M_{2,4} \qquad \text{matrice } 2 \times 4$$

$$A_{m,n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & & a_{2n} \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & & a_{mn} \end{pmatrix}$$

m, n : dimensioni della matrice

$$A_{m,n} = \left(a_{ij} \right) \qquad \begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix}$$

elementi di matrice a_{ij}

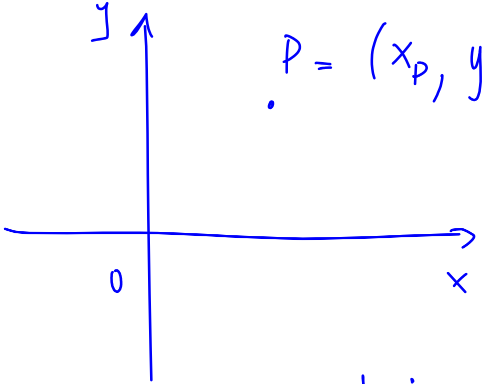
Matrici quadrate : $n = m$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ Matrice quadrata 2×2

Vettori : $m = 1$ o $n = 1$

$m = 1$ $\quad V = (v_1 , \dots , v_n)$ vettore riga

$n = 1$ $\quad W = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}$ vettore colonna



$P = (x_P, y_P)$ vettore a due componenti

Somma di matrici delle stesse dimensioni $\quad A = (a_{ij}) \quad B = (b_{ij})$

$$C = A + B = \left(a_{ij} + b_{ij} \right) = \left(c_{ij} \right)$$

$$\boxed{c_{ij} = a_{ij} + b_{ij}}$$

$$A = \begin{pmatrix} 3 & 7 \\ \frac{1}{2} & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 8 & 5 \\ 4 & \frac{3}{2} \end{pmatrix}$$

$$C = \begin{pmatrix} 3+8 & 7+5 \\ \frac{1}{2}+4 & 2+\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 11 & 12 \\ \frac{9}{2} & \frac{7}{2} \end{pmatrix}$$

Poprietà : associativa : $A + (B+C) = (A+B)+C$

commutativa : $A + B = B+A$

$\exists$ elemento neutro $\mathbb{O}$: $A + \mathbb{O} = A \quad \forall A$

$$\mathbb{O} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & & & \\ \vdots & & \cdots & \\ 0 & & & \end{pmatrix}$$

$\exists$ l'opposto $-A = (-a_{ij})$

$$A + (-A) = \mathbb{O}$$

$$a_{ij} + (-a_{ij}) = 0 \quad \forall i \ \forall j$$

Moltiplicazione di una matrice per un numero

$\lambda \in \mathbb{R} \qquad A = (a_{ij})$

$$\lambda A = (\lambda a_{ij}) = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \vdots & & & \\ \lambda a_{m1} & & \cdots & \lambda a_{mn} \end{pmatrix}$$

Prodotto di matrici

vettore riga lungo n $\qquad v = (v_1, \ldots\ v_n)$

×

vettore colonna lungo n $\qquad w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

$$V \cdot W = (V_1, \dots \; V_n) \cdot \begin{pmatrix} W_1 \\ \vdots \\ \cdot \\ W_n \end{pmatrix} = V_1 W_1 + V_2 W_2 + \dots V_n W_n$$

$$V = (2, 3) \qquad W = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

$$V \cdot W = (2, 3) \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = 2 \cdot \frac{1}{2} + 3 \cdot 1 = 4$$

Prodotto di matrici $A_{m,n} \cdot B_{p,q}$ se $\underline{n = p}$

$$C_{m,q} = A \cdot B = \left(c_{ij} = \text{prodotto tra la i-esima riga di } A \text{ e la j-esima colonna di } B \right)$$

$$A \cdot B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nq} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} b_{11} + a_{12} b_{21} \cdots + a_{1n} b_{n1} & a_{11} b_{12} + a_{12} b_{22} + \cdots a_{1n} b_{n2} & \cdots \\ a_{21} b_{11} + a_{22} b_{21} \cdots + a_{2n} b_{21} & \cdots & \\ & & \end{pmatrix}$$

$$= \left(c_{ij} \doteq \sum_{\boxed{k=1}}^{n} a_{i \underset{\uparrow}{k}} \, b_{\underset{\nearrow}{k} j} \right)$$

indice ripetuto

$$= \left(c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots \quad a_{in} b_{nj} \right)$$

$$c_{11} = a_{11} b_{11} + a_{12} b_{21} \cdots + a_{1n} b_{n1}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 7 \\ 5 & 9 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 1 \cdot 3 + 2 \cdot 5 & 1 \cdot 7 + 2 \cdot 9 \\ 3 \cdot 3 + 4 \cdot 5 & 3 \cdot 7 + 4 \cdot 9 \end{pmatrix} = \begin{pmatrix} 13 & 25 \\ 29 & 57 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

Il prodotto tra matrici NON é commutativo

$$(2,3)\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = 4$$

$A \cdot B \neq B \cdot A$ in generale

$$\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \cdot (2,3) = \begin{pmatrix} 1 & \frac{3}{2} \\ 2 & 3 \end{pmatrix} = C_{2,2}$$

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$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad w = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

$$A \cdot w = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot \frac{1}{2} + 2 \cdot 1 \\ 3 \cdot \frac{1}{2} + 4 \cdot 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{5}{2} \\ \frac{11}{2} \end{pmatrix}$$

$$w \cdot A = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

non ha $\left(\frac{1}{2}\right) \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

nemmeno senso

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} \underline{0 \quad -1} \\ 1 \quad\ \ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A \cdot B \neq B \cdot A$$

Valgono la proprietà associativa

$$(A\cdot B)\cdot C = A\cdot (B\cdot C)$$

e distributiva $\quad A\cdot (B+C) = A\cdot B + B\cdot C$

$$(B+C)\cdot A = B\cdot A \cdot C\cdot A$$

∃ l'elemento neutro $\quad \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \vdots \\ 0 & 1 & 0 & 0 & \vdots \\ 0 & 0 & 1 & 0 & \vdots \\ 0 & 0 & 0 & 1 & \\ & \cdots & & & \end{pmatrix}$

$$\mathbb{1}\cdot A = A\cdot \mathbb{1} = A \quad \forall A$$

non sempre esiste l'inversa

Data A, quando $\exists\, B \mid A \cdot B = \mathbb{1}$?

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$A \cdot B = \mathbb{0} = \begin{pmatrix} 1 \cdot 1 + 1 \cdot (-1) & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{0}$$

La matrice inversa A^{-1} di una matrice quadrata A esiste se e solo se $\det A \neq 0$

Esempio $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det A = ad - bc$

Se $\det A \neq 0$ $A^{-1} = \dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$A \cdot A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{ad-bc} =$$

$$= \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \frac{1}{ad-bc} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} ae + \cancel{bg} & \cancel{af} + bh \\ ce + \cancel{dg} & \cancel{cf} + dh \end{pmatrix}$$

$$B = \mathbb{1} \qquad e = h = 1 \qquad f = g = 0$$

$$A \cdot B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

Riduzione di una matrice (metodi Gauss)

Data una matrice A qualunque

facciamo le seguenti operazioni:

i) moltiplichiamo tutti gli elementi di una riga per una stessa costante $\lambda \neq 0$

ii) scambiamo due righe

iii) sommiamo (termine a termine) a una riga un'altra riga moltiplicata per una costante $\lambda \neq 0$

Si può ridurre una matrice alla forma

$$\begin{pmatrix} a & e & f & h & \\ 0 & b & g & & \cdots \\ 0 & 0 & c & & \\ 0 & 0 & 0 & d & \\ \cdot & & & & \end{pmatrix}$$

cioè con solo 0 sotto la diagonale

Una matrice quadrata della forma

$$T = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \\ 0 & a_{22} & a_{23} & \cdots \\ 0 & 0 & a_{33} & \end{pmatrix}$$

$$0 \quad 0 \quad 0$$

si dice matrice triangolare e il suo determinante è il prodotto degli elementi sulla diagonale

$\det T = a_{11}\, a_{22}\, a_{33} \dots$

iii) non cambia il determinante

ii) cambia il determinante di un segno

i) moltiplica il determinante per λ

Per calcolare l'inversa di una matrice quadrata A si costruisce la matrice rettangolare $X = \left(A \,\vdots\, \mathbb{1} \right)$, poi si usano le operazioni i), ii), iii) per ridurla alla forma

$$\left(\mathbb{1} \;\vdots\; A' \right)$$ Allora A' è l'inversa di A

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix}$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Voglio arrivare a $X' = \begin{pmatrix} 1 & 0 & a' & b' \\ 0 & 1 & c' & d' \end{pmatrix}$

Supponiamo $\det A \neq 0$ $\quad \underline{\det A = ad - bc}$

Cioè $a \neq 0$ o $c \neq 0$

A meno di scambiare le righe supponiamo $a \neq 0$

$$X = \begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \qquad a \neq 0$$

i) moltiplico la prima riga per $\frac{1}{a}$

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{pmatrix}$$

iii) sottraggo alla 2^a riga la prima moltiplicata per c

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \boxed{d - \frac{bc}{a}} & 0 - \frac{c}{a} & 1 \end{pmatrix}$$

i) moltiplico la 2^a per $\dfrac{1}{d - \frac{bc}{a}}$

$d \neq \frac{bc}{a}$ $\qquad ad - bc \neq 0$

$$\begin{pmatrix} 1 & \boxed{\frac{b}{a}} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-\frac{c}{a}}{d-\frac{bc}{a}} & \frac{1}{d-\frac{bc}{a}} \end{pmatrix}$$

$$1^A \to 1^A - 2^A \times \frac{b}{a}$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{a} - \frac{b}{a}\,\frac{-\frac{c}{a}}{d-\frac{bc}{a}} & -\frac{b}{a}\,\frac{1}{d-\frac{bc}{a}} \\ 0 & 1 & \frac{-\frac{c}{a}}{d-\frac{bc}{a}} & \frac{a}{a\left(d-\frac{bc}{a}\right)} \end{pmatrix}$$

$$\frac{1}{a} - \frac{b}{a} \, \frac{-\frac{c}{a}}{d - \frac{bc}{a}} = \frac{1}{a} + \frac{bc}{a \cdot a \cdot \left(d - \frac{bc}{a}\right)} =$$

$$= \frac{1}{a} + \frac{bc}{a\,(ad - bc)} = \frac{ad - \cancel{bc} + \cancel{bc}}{a\,(ad - bc)} =$$

$$= \frac{d}{\det A}$$

$$A^{-1} = \begin{pmatrix} \frac{d}{\det A} & \frac{-b}{\det A} \\ \frac{-c}{\det A} & \frac{a}{\det A} \end{pmatrix}$$

<u>ok!</u>

Applicazioni : sistemi di equazioni lineari

incognite $x_1, \dots x_n$

una equazione lineare è

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

dove $a_1, \dots a_n$ e b sono coefficienti dati

Sistema di m equazioni

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2 \\ \dots \qquad\qquad\qquad \dots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m \end{cases}$$

$$AX = B$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & & \cdots & a_{mn} \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Esercizio

$$\begin{cases} x + 2y - 4z = 0 \\ 3x + 0y + 6z = -1 \\ 4x + 4y - 6z = 1 \end{cases}$$

$$AX = b \qquad A = \begin{pmatrix} 1 & 2 & -4 \\ 3 & 0 & 6 \\ 4 & 4 & -6 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$A \cdot X = \left(\begin{array}{ccc} 1 & 2 & -4 \\ \hline 3 & 0 & 6 \\ \hline 4 & 4 & -6 \end{array}\right) \left(\begin{array}{|c} x \\ y \\ z \end{array}\right) =$$

$$= \begin{pmatrix} 1 \cdot x + 2y - 4z \\ 3 \cdot x + 0 \cdot y + 6z \\ 4 \cdot x + 4y - 6z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = b$$

$$C = \left(\begin{array}{c|c} A & b \end{array}\right) \longrightarrow \left(\begin{array}{c|c} \mathbb{1} & \begin{matrix} s \\ o \\ l \\ u \\ z \\ i \\ o \\ n \\ i \end{matrix} \end{array}\right)$$

$$C = \begin{pmatrix} 1 & 2 & -4 & 0 \\ 3 & 0 & 6 & -1 \\ 4 & 4 & -6 & 1 \end{pmatrix}$$

$$2^1 \longrightarrow 2^1 - 3 \cdot 1^1$$

$$\begin{pmatrix} 1 & 2 & -4 & 0 \\ 0 & -6 & 18 & -1 \\ 4 & 4 & -6 & 1 \end{pmatrix}$$

$$3^1 \to 3^1 - 4 \cdot 1^1$$

$$-4 + x \cdot (-6) = 0$$

$$\begin{pmatrix} 1 & 2 & -4 & 0 \\ 0 & \textcircled{-6} & 18 & -1 \\ 0 & -4 & 10 & 1 \end{pmatrix}$$

$$3^A \to 3^A - \frac{2}{3}\,2^A$$

$$-4+\frac{2}{\cancel{3}}\left(-4\cancel{6}\right)^{2}=0$$

$$10-\frac{2}{\cancel{3}}\cancel{18}^{6}=-2$$

$$1-\frac{2}{3}(-1)=\frac{5}{3}$$

$$\boxed{\det A = 12}$$

$$\neq 0$$

$$\begin{pmatrix} 1 & 2 & \textcircled{-4} & 0 \\ 0 & -6 & 18 & -1 \\ 0 & 0 & \textcircled{-2} & \frac{5}{3} \end{pmatrix}$$

$$1^A \to 1^A - 2\cdot 3^A$$

$$0-2\cdot\frac{5}{3}$$

$$\begin{pmatrix} 1 & 2 & 0 & -\frac{10}{3} \\ \underline{0} & -6 & 18 & -1 \\ 0 & 0 & -2 & 5/3 \end{pmatrix}$$

$$2^A \to 2^A + 9\cdot 3^A$$

$$-1+{}^{3}\cancel{9}\cdot\frac{5}{\cancel{3}}=14$$

$$\begin{pmatrix} 1 & 2 & 0 & -\frac{10}{3} \\ 0 & -6 & 0 & 14 \\ 0 & 0 & -2 & 5/3 \end{pmatrix}$$

$$1^A \longrightarrow 1^A + \frac{1}{3} \cdot 2^A$$

$$\begin{pmatrix} 1 & 0 & 0 & 4/3 \\ 0 & -6 & 0 & 14 \\ 0 & 0 & -2 & 5/3 \end{pmatrix}$$

$$-\frac{10}{3} + \frac{1}{3} 14 = \frac{4}{3}$$

$$2^A \longrightarrow -\frac{1}{6} 2^A$$

$$3^A \longrightarrow -\frac{1}{2} 3^A$$

$$\begin{pmatrix} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & -\frac{7}{3} \\ 0 & 0 & 1 & -\frac{5}{6} \end{pmatrix}$$

$$x = \frac{4}{3} \qquad y = -\frac{7}{3} \qquad z = -\frac{5}{6}$$

$$x + 2y - 4z = 0$$
$$3x + 0y + 6z = -1$$
$$4x + 4y - 6z = \underline{1}$$

$$\frac{4}{3} - \frac{7}{3} \cdot 2 - \overset{2}{\cancel{4}} \frac{-5}{\cancel{6}_{3}} = \frac{4 - 14 + 10}{3}$$

$$\cancel{3} \cdot \frac{4}{\cancel{3}} + \cancel{6} \frac{-5}{\cancel{6}} = -1$$

$$4 \cdot \frac{4}{3} + 4 \cdot \frac{-7}{3} - \overset{3}{\cancel{6}} \cdot \left(-\frac{5}{\underset{3}{\cancel{6}}}\right) = \frac{16-28+15}{3} = 1$$

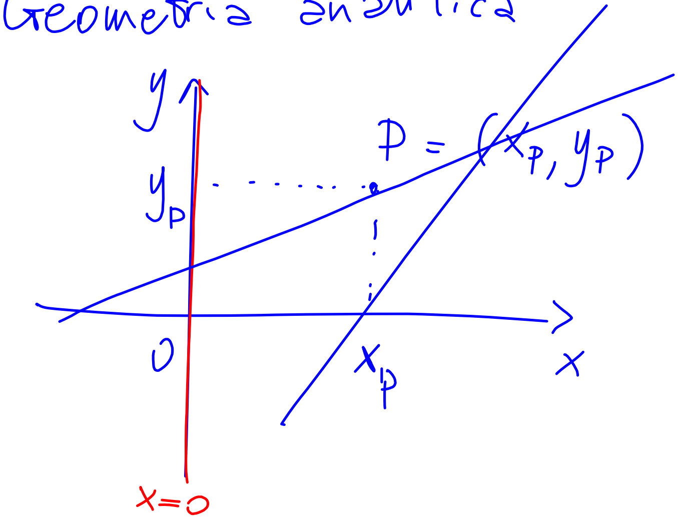
$$AX = b \qquad \det A \neq 0$$

moltiplico entrambi per A^{-1} $\qquad \exists\, A^{-1}$

$$A^{-1}AX = A^{-1}b \qquad A^{-1}A = \mathbb{1}$$

$$X = A^{-1}b$$

Geometria analitica



Retta: luogo geometrico dei punti (x,y) che soddisfano un'equazione lineare

$$ax + by + c = 0$$

lineare: solo x e y, NO x^2, y^2, xy, x^3, x^2y

solo monomi di grado 1 o zero nelle incognite

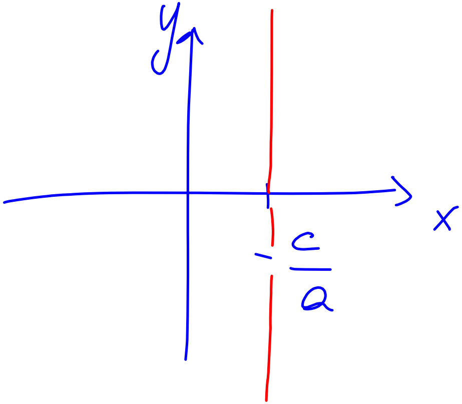
$x=0$ $(a=1,\ b=c=0)$ l'asse y

$y=0$ è l'asse x

$y = f(x)$ $\qquad$ $ax + by + c = 0$

$b = 0$: $\qquad$ $ax + c = 0$

$a \neq 0$ $\qquad$ $x = -\dfrac{c}{a}$



$b \neq 0$ $\qquad$ $ax + c = -by$

$$y = -\frac{a}{b}x - \frac{c}{b} = mx + q$$

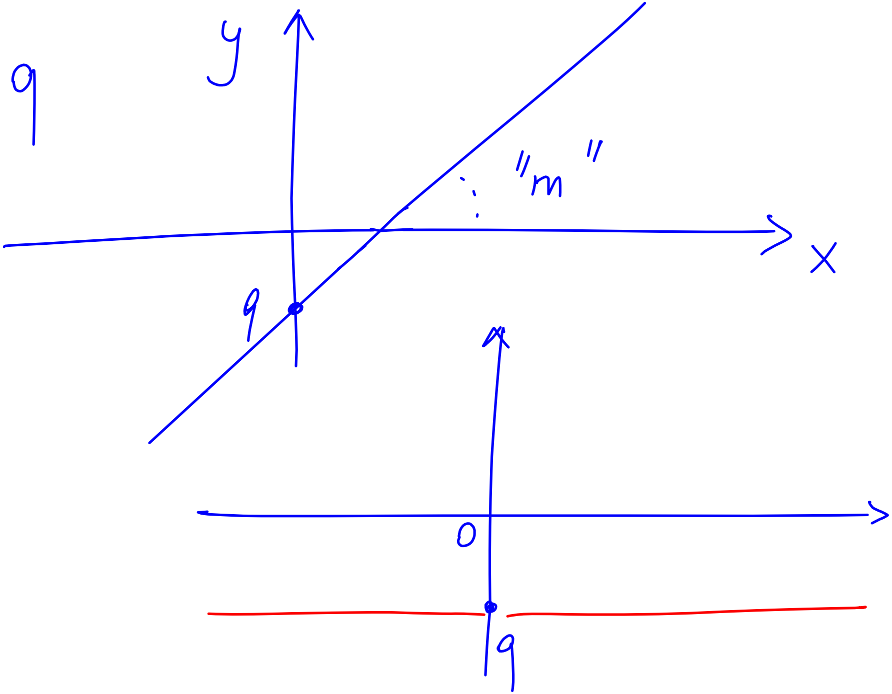
$m = -\dfrac{a}{b}$ si chiama coefficiente angolare

$q = -\dfrac{c}{b}$ si chiama termine noto

$$y = mx + q$$

$$x = 0 \Rightarrow y = q$$

$$m = 0 \qquad y = q$$



Esercizio : rette parallele

$$\begin{cases} ax + by + \underline{\underline{c}} = 0 \\ a'x + b'y + \underline{\underline{c'}} = 0 \end{cases} \qquad A = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \qquad C = \begin{pmatrix} -c \\ -c' \end{pmatrix}$$

$$\boxed{AX = C} \qquad \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -c \\ -c' \end{pmatrix}$$

Le rette sono parallele quando il sistema NON ha soluzioni

Se $\exists A^{-1}$ allora $\underbrace{A^{-1} A}_{} X = A^{-1} C = X$

$(\det A \neq 0)$

NON ci sono soluzioni $\Longleftrightarrow$ $\det A = 0$

$\Longrightarrow$ implica

$\Longleftarrow$ è implicato da

$\Longleftrightarrow$ se e solo se

$$\det A = \det \begin{pmatrix} \textcircled{a} & b \\ a' & \textcircled{b'} \end{pmatrix} = ab' - a'b = 0$$

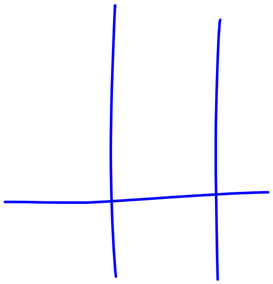
$$\underline{ab' = a'b}$$

$b \neq 0$ $b' \neq 0$: $m = -\dfrac{a}{b} = -\dfrac{a'}{b'} = m'$

$b = 0 \Rightarrow \quad a b' = 0 \quad a \neq 0 \Rightarrow b' = 0$

$ax + c = 0$ retta verticale $a \neq 0$

$a'x + c' = 0$ retta verticale



$$m = -\frac{a}{b} = m' = -\frac{a'}{b'}$$

$b = 0 \Rightarrow m = \infty$

$$\boxed{\frac{1}{0} = \infty}$$

"∞" = infinito

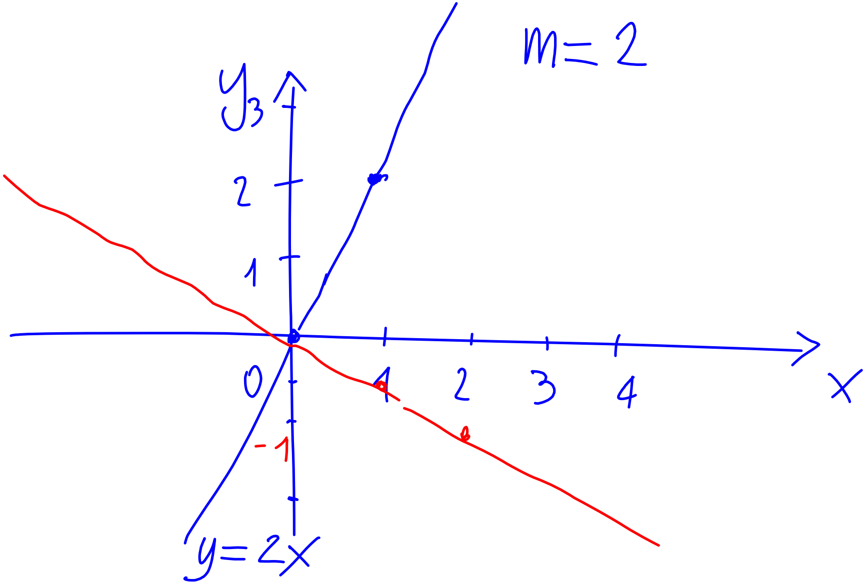
$$\frac{1}{\frac{1}{10^6}} = 10^6 \qquad 10^6 = 1\,000\,000 = 1 \text{ milione}$$

Due rette sono perpendicolari

quando $m' = -\frac{1}{m}$

Esempio $y = 2x$ $y = -\frac{x}{2}$

$m = 2$ $m' = -\frac{1}{2} = -\frac{1}{m}$



Equazioni quadratiche $abc \neq 0$

$$ax^2+by^2+cxy+dx+ey+f=0$$

combinazione lineare di x^2, y^2, xy, x, y, 1

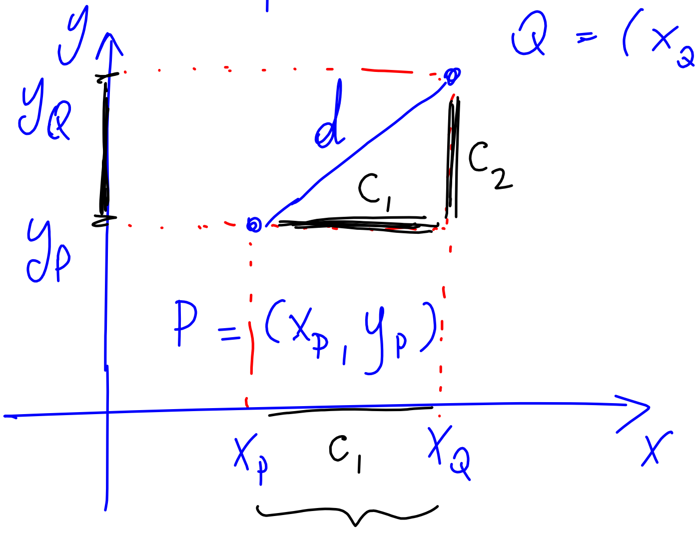
= somma con coefficienti numerici non specificati

Le soluzioni sono: circonferenze, ellissi, parabole, iperboli

Circonferenza: il luogo geometrico dei punti (x, y) del piano che hanno uguale distanza R

da un punto dato (x_c, y_c) del piano

$Q = (x_2, y_2)$



$$d_{PQ} = \sqrt{c_1^2 + c_2^2}$$

$$c_1 = x_Q - x_P$$

$$c_2 = y_Q - y_P$$

$$d_{PQ} = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}$$

$$R = \sqrt{(x - x_c)^2 + (y - y_c)^2}$$

$$(a+b)^2 = (a+b)(a+b) =$$
$$= a^2 + b^2 + 2ab$$

elevo al quadrato :

$$\boxed{R^2 = (x - x_c)^2 + (y - y_c)^2} =$$

$$= \underline{\underline{x^2}} + \underbrace{x_c^2} - \underbrace{2x x_c} + \underline{y^2} + \underbrace{y_c^2} - \underbrace{2y y_c}$$

$$\underline{\underline{x^2}},\ \underline{\underline{y^2}},\ \underset{\times}{xy},\ \underline{\underline{x}},\ \underline{\underline{y}},\ \underset{w}{1}$$

$$x^2 + y^2 - 2x x_c - 2y y_c + (x_c^2 + y_c^2 - R^2) = 0$$

R = raggio (x_c, y_c) = centro

Esercizio : $x^2+y^2-2x-4=0$

è una circonferenza ? Se sì , trovare raggio e centro

La circonferenza centrata nell'origine

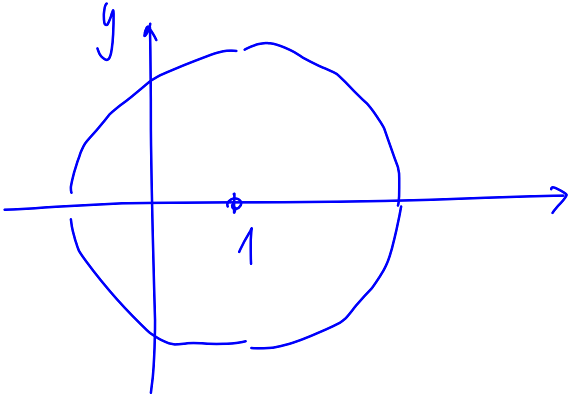
$x_c = 0, \ y_c = 0$ $\qquad \underline{R^2 = x^2+y^2}$

$x^2+y^2-2x = 4$ $\qquad (x-\overset{1}{\underline{p}})^2+\underline{y}^2 = \overset{5}{q} = R^2$

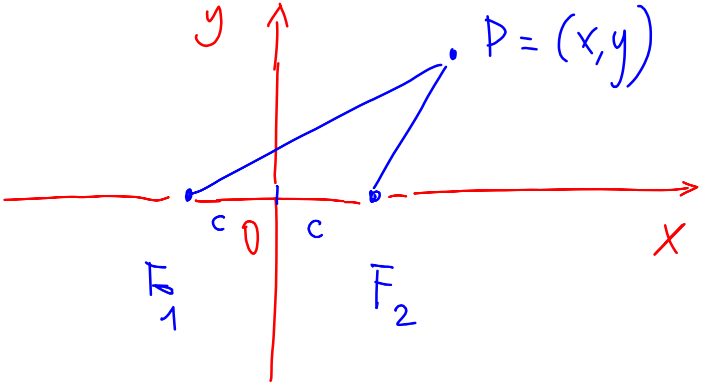
$\underline{p=1}$ $\quad q = 5$ $\qquad x^2+p^2-2px+y^2 = q$

$q-p^2 = 4$ $\quad R = \sqrt{5}$ $\qquad x^2+y^2-\underline{\underline{2p}}x = q-p^2$

centro $(1, 0)$



Ellisse: insieme dei punti $P = (x, y)$ tali che é costante $(= 2a)$ la somma delle distanze tra P e due punti dati F_1 e F_2 (fuochi)



$P : (x,y)$

$d_{PF_1} + d_{PF_2} = 2a$

$F_1 = (-c, 0) \qquad F_2 = (c, 0)$

$$d_{PF_2} = \sqrt{(x-c)^2 + y^2}$$

$$d_{PF_1} = \sqrt{(x+c)^2 + y^2}$$

$$2a = \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2}$$

$$2a - \sqrt{(x-c)^2+y^2} = \sqrt{(x+c)^2+y^2}$$

elevo al quadrato :

$$4a^2 + (x-c)^2 + \cancel{y^2} - 4a\sqrt{(x-c)^2+y^2} = (x+c)^2 + \cancel{y^2}$$

$$4a^2 + \cancel{x^2} + \cancel{c^2} - 2cx - 4a\sqrt{(x-c)^2+y^2} = \cancel{x^2} + \cancel{c^2} + 2cx$$

$$\cancel{4}a^2 - \cancel{4}cx = \cancel{4}a\sqrt{(x-c)^2+y^2}$$

$$a^2 - cx = a\sqrt{(x-c)^2+y^2}$$

elevo al quadrato

$$a^2 - cx = a\sqrt{(x-c)^2+y^2}$$

$$a^4 + c^2x^2 - 2a^2cx = a^2\left((x-c)^2+y^2\right)$$

$$a^4 + c^2x^2 - 2a^2cx = a^2\left(x^2+c^2-2cx+y^2\right)$$

$$a^4 + c^2x^2 - \cancel{2a^2cx} = a^2x^2 + a^2c^2 - \cancel{2a^2cx} + \underline{a^2y^2}$$

$$0 = -a^4 - \underline{c^2x^2} + \underline{a^2x^2} + a^2c^2 + \underline{a^2y^2}$$

$$= \underline{(a^2-c^2)}x^2 + \underline{a^2y^2} - \underline{a^4 + a^2c^2}$$

$$\underline{a^4 -} a^2c^2 = b^2x^2 + a^2y^2 = a^2(a^2-c^2) = a^2b^2$$

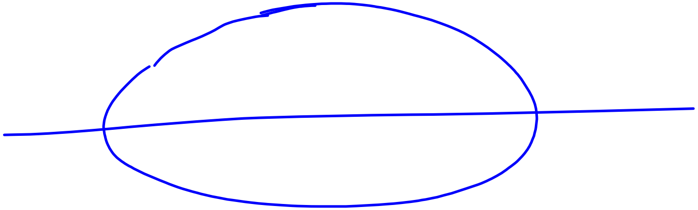
$$b^2 = a^2 - c^2$$

$$\frac{\cancel{b^2}x^2}{a^2\cancel{b^2}} + \frac{\cancel{a^2}y^2}{\cancel{a^2}b^2} = \frac{\cancel{a^2}\cancel{b^2}}{\cancel{a^2}\cancel{b^2}}$$

divido per a^2b^2

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$ ellisse

circonferenza : $a = b = R$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$ iperbole

Iperbole: insieme dei punti $P = (x, y)$ tali che è costante la differenza delle distanze tra P e due punti dati F_1 e F_2 (fuochi)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

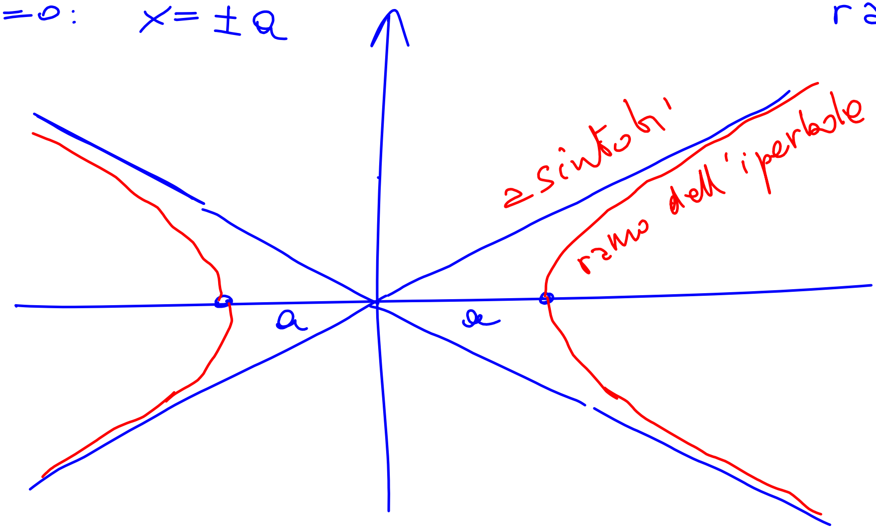
Considero $\frac{x^2}{a^2} = \frac{y^2}{b^2}$

radice $\frac{x}{a} = \frac{y}{b}$

$\frac{x}{a} = -\frac{y}{b}$

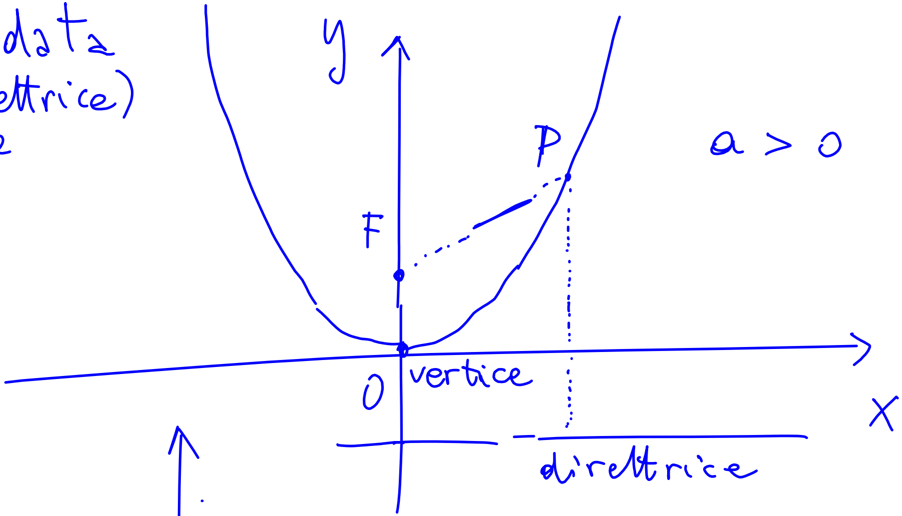
asintoti

$y=0$: $x = \pm a$



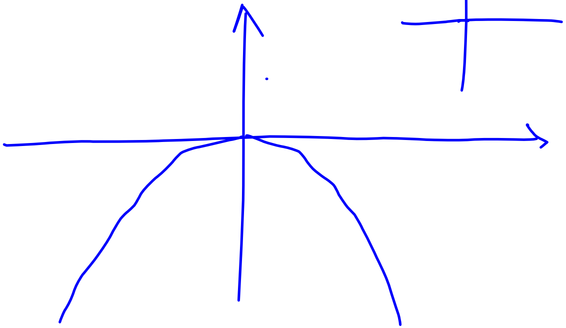
Parabola : luogo geometrico dei punti P= (x,y) tali che è costante la somma delle distanze tra P e un punto dato F (fuoco) + P e una retta data (direttrice)

$$y = a x^2$$

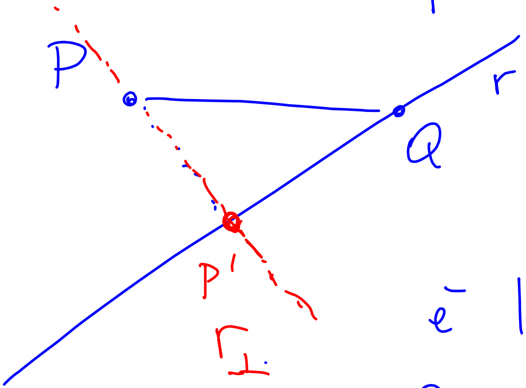


$a > 0$

$a < 0$



Distanza tra un punto e una retta



d_{PQ}

è la distanza minima tra P e i punti Q della retta

Si cerca la retta $r_\perp$ perpendicolare ad r che passa per P

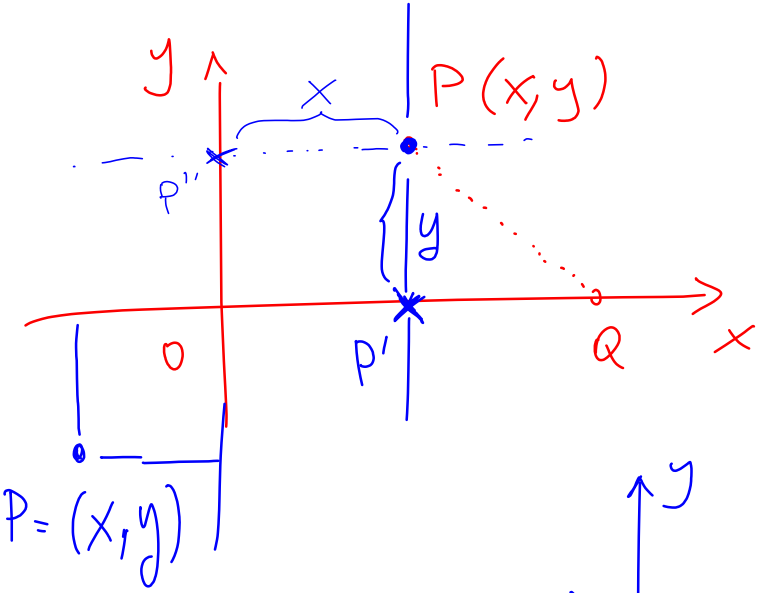
$\perp$ = perpendicolare

r e $r_\perp$ si intersecano in un punto P'

$d_{PP'}$ = distanza tra P e la retta r

Trovare la distanza tra $P = (x, y)$ e l'asse x

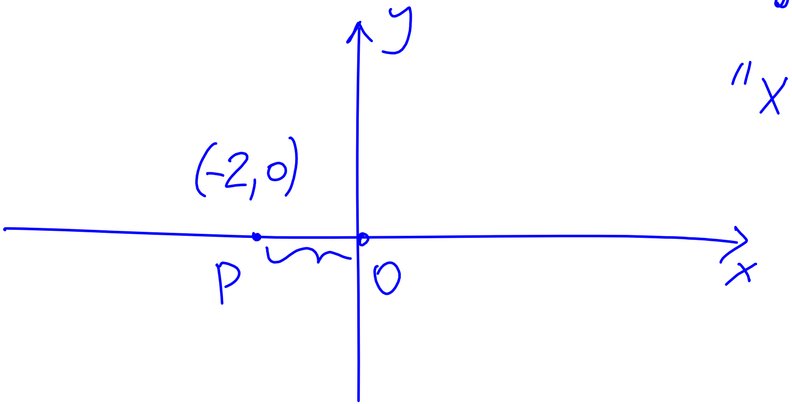
è $= |y|$



$|x|$ = distanza tra P e l'asse y

$|x|$ = valore assoluto di x = modulo di x

"x senza il segno"



Funzioni

$f : D \longrightarrow \mathbb{R}$

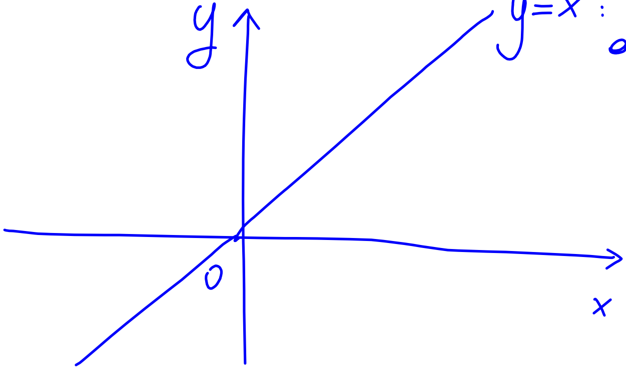
$D \subset \mathbb{R}$ dominio

x : variabile indipendente

y : variabole dipendente

$y = f(x)$

$x \in D$ $y \in \mathbb{R}$



$y=x$: retta diagonale

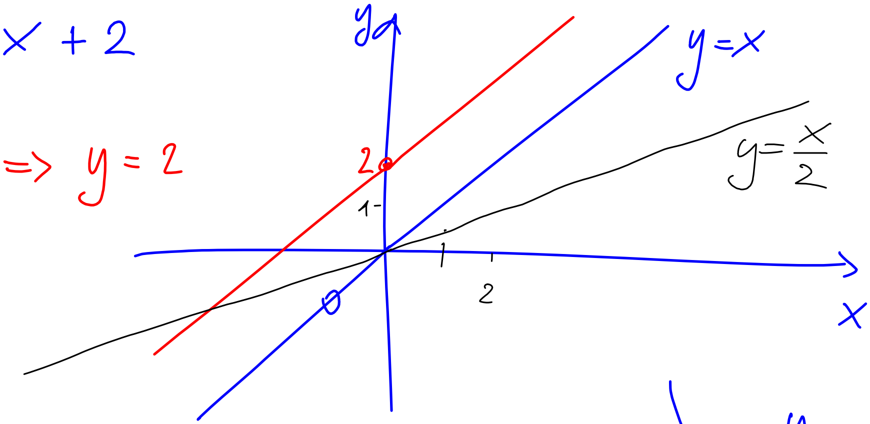
L'insieme dei punti con coordinate $(x, f(x))$ si dice grafico della funzione

$y = x$: insieme dei punti (x, x)

$y = x$ è definita su tutto $\mathbb{R}$ (cioè posso prendere $D = \mathbb{R}$)

$y = x + 2$

$x = 0 \Rightarrow y = 2$

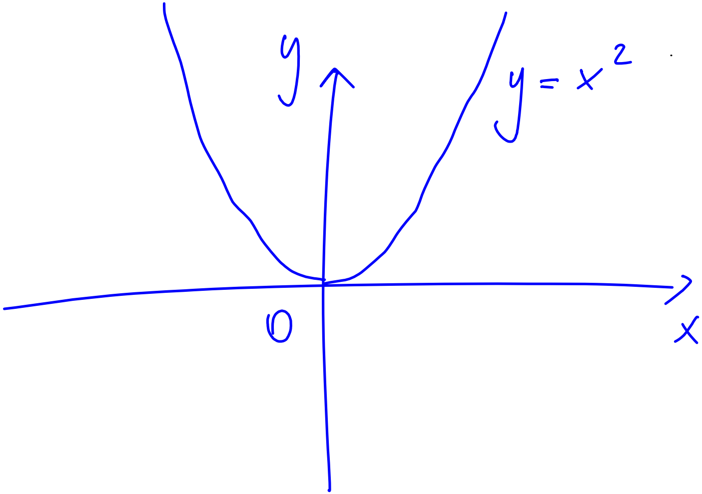


$y = \frac{x}{2}$

Anche qui $D = \mathbb{R}$

Parabola $y = x^2$

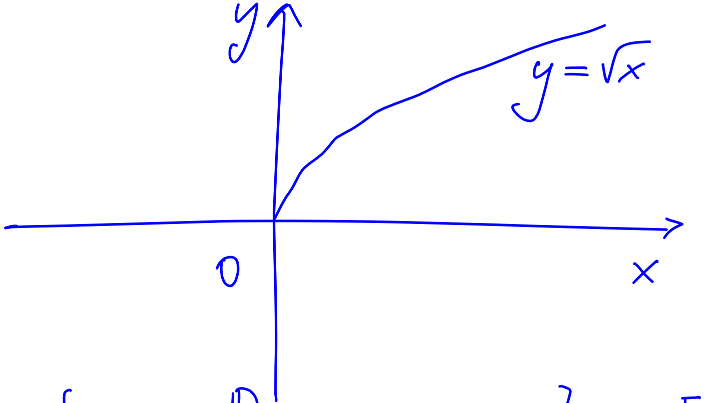
$D = \mathbb{R}$



$y = \sqrt{x}$ $\quad D = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$

$f : D \longrightarrow \mathbb{R}$

$\sqrt{-1}$ non è reale



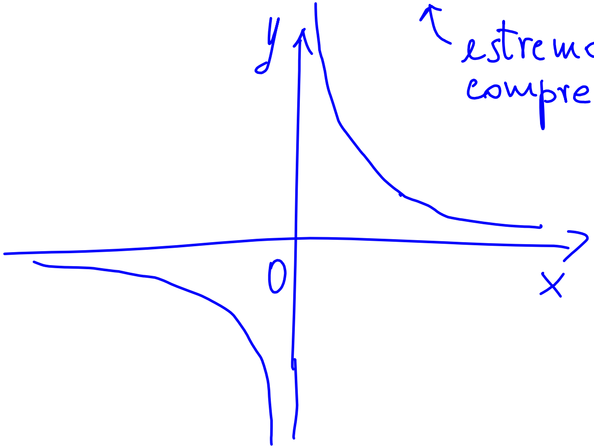
$y = \sqrt{x-3}$ $\quad D = \{x \in \mathbb{R} : x \geq 3\} = [3, \infty)$

estremo escluso

estremo compreso

$y = \frac{1}{x}$ $\quad D = \mathbb{R} \setminus \{0\}$

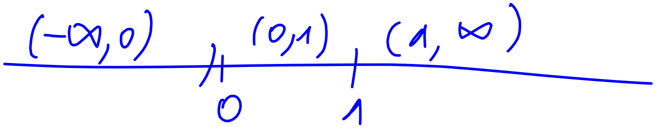
iperbole



$$y = \frac{1}{x(x-1)} \qquad D = \mathbb{R} \setminus \{0, 1\} = (-\infty, 0) \cup (0,1) \cup (1, \infty)$$

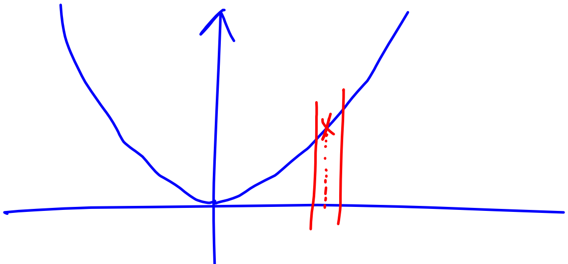
$$= \frac{1}{x} \cdot \frac{1}{x-1}$$

$\frac{1}{0}$ non ha senso



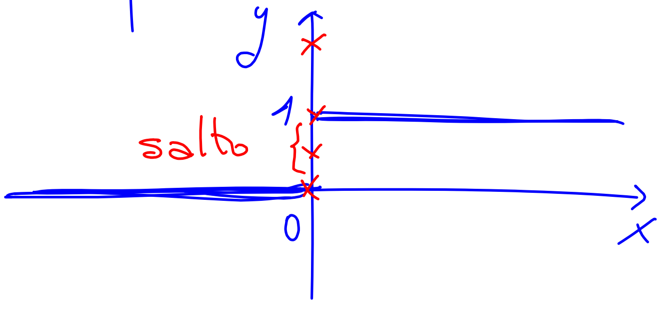
Continuità

"non ci sono interruzioni"



funzione discontinua:

$$y = \begin{cases} 1 & \text{per } x > 0 \\ 0 & \text{per } x < 0 \end{cases}$$

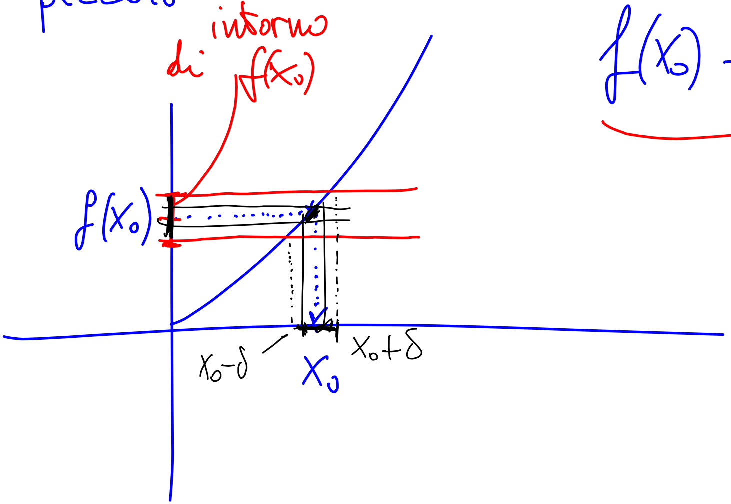


Una funzione $f: D \longrightarrow \mathbb{R}$ si dice continua in $x_0 \in D$ se

$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad : \quad \forall x \in (x_0 - \delta, x_0 + \delta)$

"arbitrariamente piccolo"

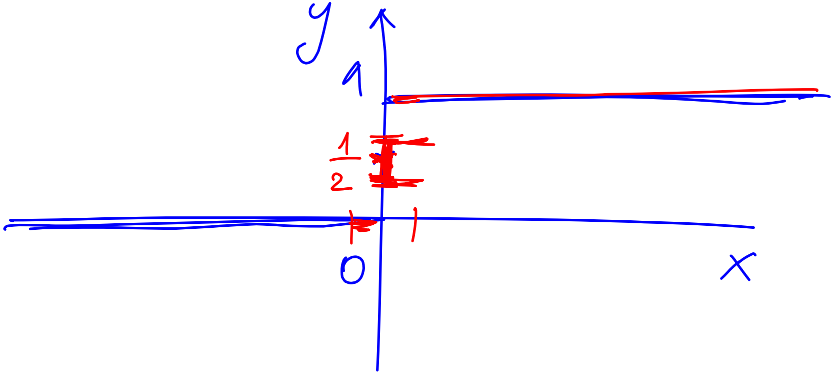
"intorno di x_0"



$$\underbrace{f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon}_{\text{intorno di } f(x_0)}$$

$$(x_0, f(x_0))$$

Intorno di un punto P = intervallo aperto $I = (a, b) : P \in I$



$$y = \begin{cases} 1 & \text{per } x > 0 \\ \frac{1}{2} & \text{per } x = 0 \\ 0 & \text{per } x < 0 \end{cases}$$

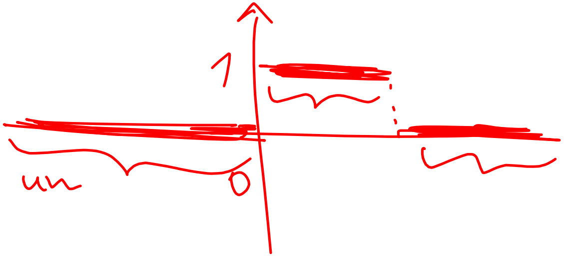
Considero $x_0 = 0$ $f(x_0) = \frac{1}{2}$

non cambia niente

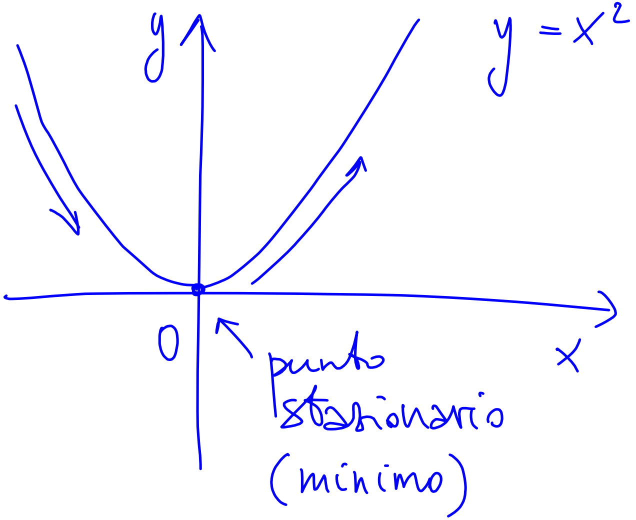
$$y = \begin{cases} 1 & \text{per } x > 0 \\ \alpha & \text{per } x = 0 \\ 0 & \text{per } x < 0 \end{cases}$$

Funzione continua a tratti:

funzione continua tranne che in un numero finito di punti



Funzioni crescenti o decrescenti



crescente per $x > 0$

decrescente per $x < 0$

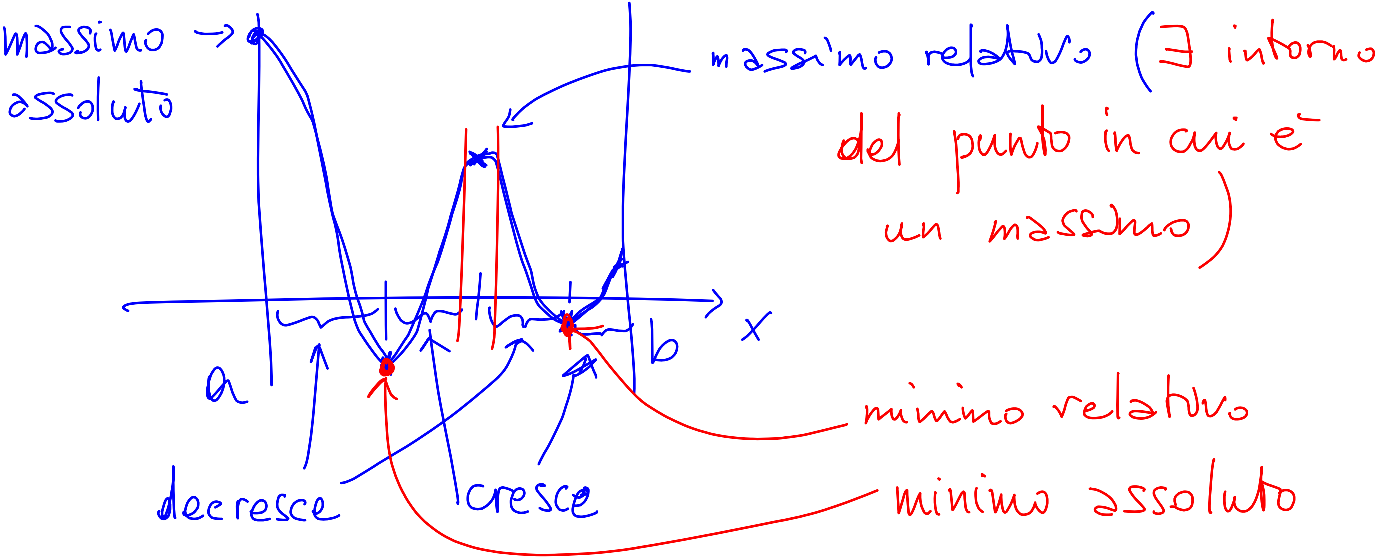
$f(x)$ è crescente in $[a, b]$ se $\forall x_1, x_2 \in [a, b]$

tali che $x_1 < x_2$ vale $f(x_1) < f(x_2)$

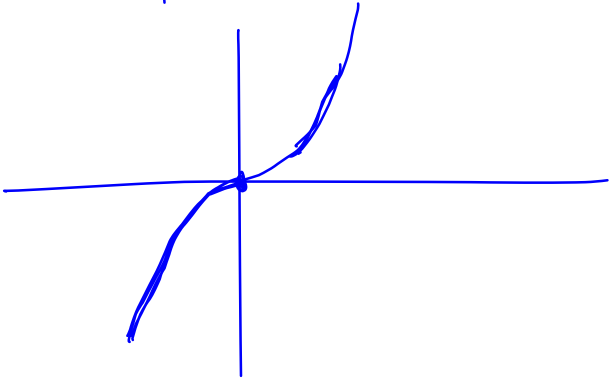
$f(x)$ è decrescente se $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

I punti stazionari sono i punti in cui la funzione non cresce e non decresce

[∄ intorno del punto in cui cresce o decresce]



Altro tipo di punti stazionari : flesso



Studieremo funzioni polinomiali, esponenziali, logaritmiche, trigonometriche e operazioni fra funzioni: algebriche, composizione e inversa

$y = f(x)$ $\qquad f: D_f \longrightarrow \mathbb{R}$

$y = g(x)$ $\qquad g: D_g \longrightarrow \mathbb{R}$

$y = f(x) + g(x)$ $\qquad f+g : D_f \cap D_g \longrightarrow \mathbb{R}$

$y = f(x)\, g(x)$ $\qquad f \cdot g : D_f \cap D_g \longrightarrow \mathbb{R}$

$y = \dfrac{f(x)}{g(x)}$ $\qquad \dfrac{f}{g} : D_f \cap D_g \setminus \{x : g(x) = 0\} \longrightarrow \mathbb{R}$

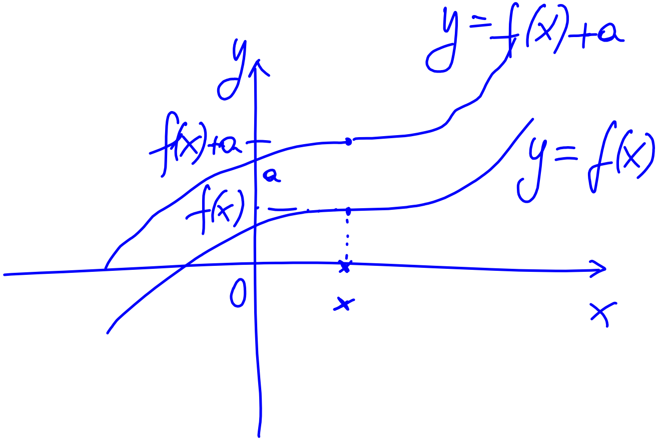
Esempio : $f(x) = x \quad g(x) = \dfrac{1}{x}$

$f(x) \cdot g(x) = x \cdot \dfrac{1}{x} = 1$ definita in $\mathbb{R} \setminus \{0\}$

$g(x) = a =$ costante

$y = f(x) + g(x) = f(x) + a$

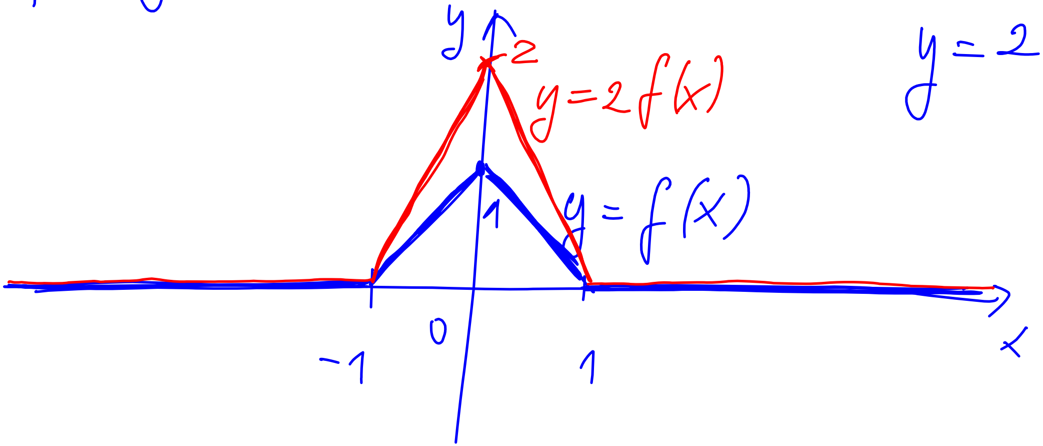
traslazione di a lungo y



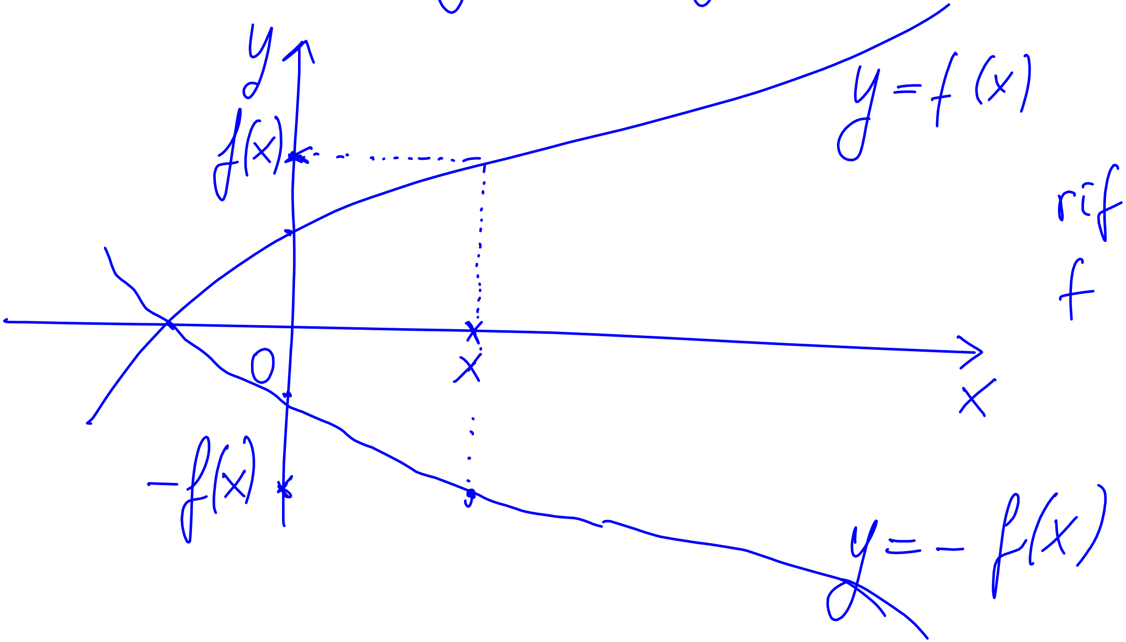
$y = f(x)g(x) = a\,f(x)$

$a = 2$

$y = 2\,f(x)$



$a = -1$ $\quad y = f(x)\,g(x) = -f(x)$



riflessione di f rispetto all'asse x

Composizione di funzioni:

$f : D_f \longrightarrow \mathbb{R} \qquad g : D_g \longrightarrow \mathbb{R} \qquad f \circ g$

$y = f(x) \qquad y = g(x)$

$y = f(g(x))$

Esempio: $\quad y = f(x) = \sqrt{x} \qquad y = g(x) = \underline{x+2}$

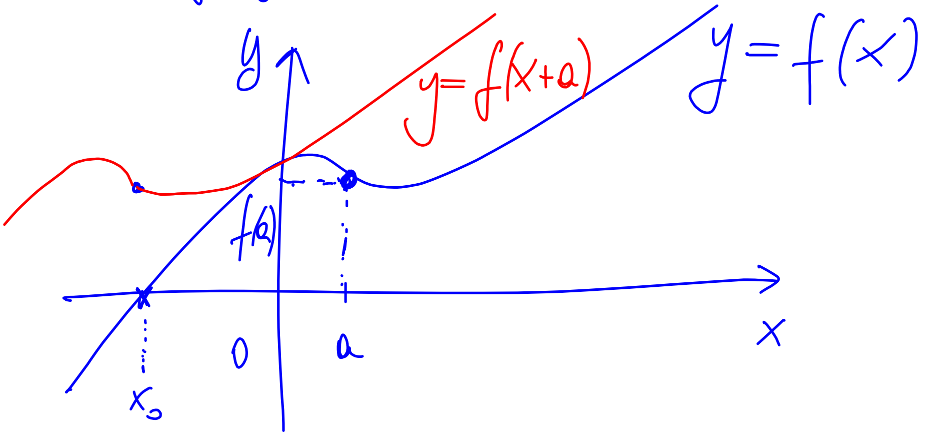
$y = f(g(x)) = \sqrt{x+2}$

$f \circ g \neq g \circ f \qquad g(f(x)) = \sqrt{x} + 2$

Esempio: $y=f(x)$ $\quad y=g(x)=x+a$

a = costante

$y=f(g(x))=f(x+a)$ traslazione lungo x di $-a$



$a=2$

$f(x) \qquad f(x+2)$

$\uparrow \qquad\qquad \uparrow$

$0 \; \ldots\ldots \; -2 \ldots f(0)$

$1 \; \ldots\ldots \; -1 \ldots f(1)$

$f(x_0)=0 \qquad f(x_0+a)=f(a)$

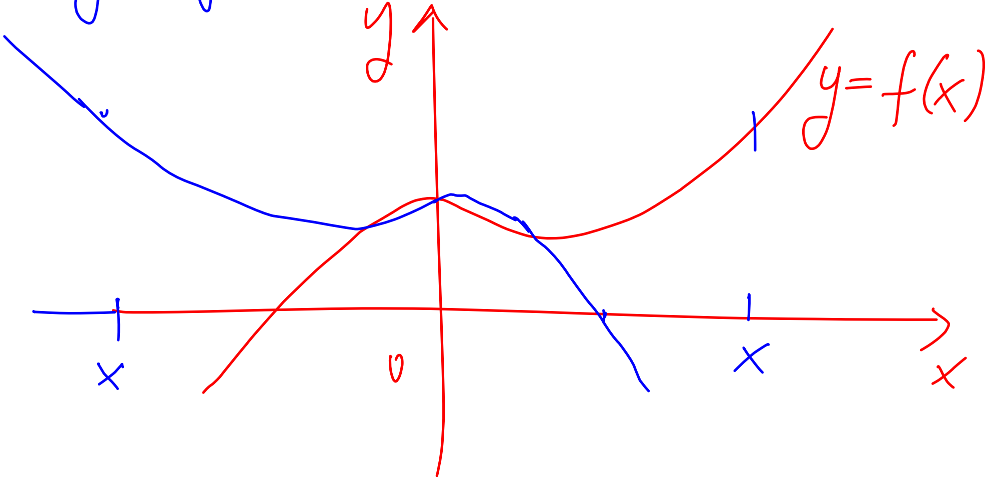
$f(x+a)=f(x')$ sono uguali in punti

$x'=x+a$

$g(x) = \underline{\underline{-x}}$ $\qquad y = f(g(x)) = f(-x)$

$y = f(x)$

riflessione rispetto a y



$y = f(x) = f(-x')$ dove? per $x' = -x$

Funzione inversa $y = f(x)$

cerco di invertire la funzione se possibile risolvendo x in funzione di y :

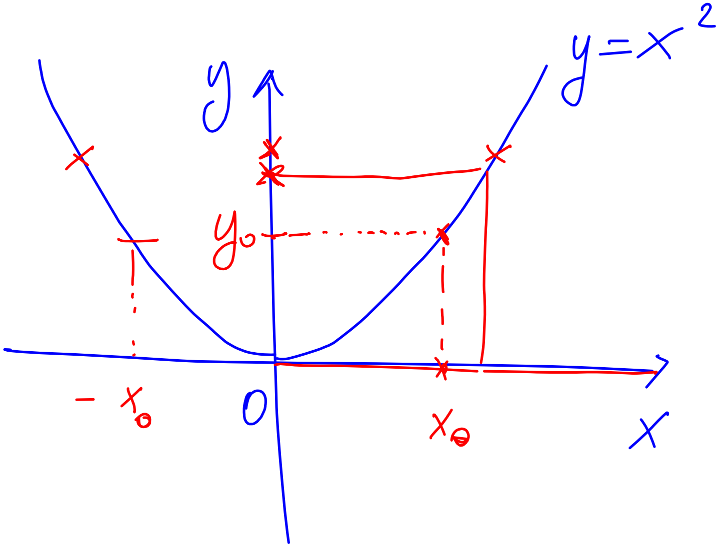
$x = f^{-1}(y)$ "f^{-1} = funzione inversa"

Poi ridefinite x come y e y come x

$y = f^{-1}(x)$

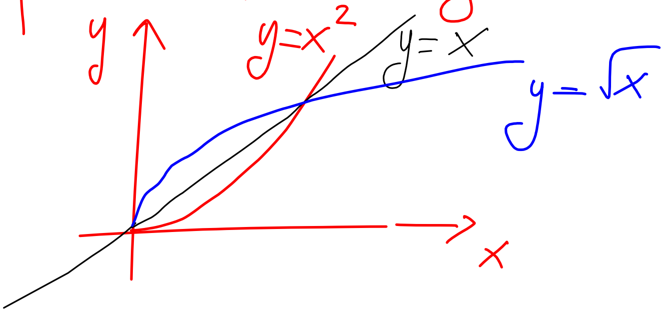
Esempi : $y = x^2$ $x = \sqrt{y}$ $\underline{y > 0}$

Scambio : $y = \sqrt{x}$ $x > 0$



ridurre il dominio da $\mathbb{R}$ a $\mathbb{R}_+$

Il grafico della funzione inversa è il riflesso rispetto alla diagonale $y=x$ (infatti, scambia $x \leftrightarrow y$)



Funzioni elementari

potenze $\overset{\text{esponente} \to}{} \quad \underset{\uparrow \text{base}}{a}^{b} = \underbrace{a \cdot a \cdot a \cdots a}_{b \text{ volte}}$ (se $b \in \mathbb{N}$)

ha senso se

$a > 0 \quad b \in \mathbb{R}$

$a = 0 \quad b > 0$

$a < 0 \quad b = \frac{m}{n} \quad m, n \in \mathbb{Z}$, n dispari

$a^{\frac{1}{n}}$ = radice n.esima (cioè quel c tale che $c^n = a$)

$a < 0$: $a^{\frac{1}{n}}$ ha senso solo per n = dispari

$(-8)^{\frac{1}{2}}$ non ha senso

$(-8)^{\frac{1}{3}} = -2$ $\qquad \underbrace{(-2)\cdot(-2)\cdot(-2)}_{\text{tre volte}} = -8$

$a > 0 \qquad a^{\frac{1}{n}} \qquad n \in \mathbb{N}$

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}}$$

Proprietà $\qquad a^{b+c} = a^b \cdot a^c$

$$2^{3+7} = 2^{10} = \underbrace{2\cdot 2 \cdot \cdot \cdot \quad 2}_{\text{10 volte}} =$$

$$= \underbrace{(2\cdot 2\cdot 2)}_{\text{3 volte}} \cdot \underbrace{(2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2)}_{\text{7 volte}} = 2^3 \cdot 2^7$$

$$a^{b\cdot c} = (a^b)^c = (a^c)^b$$

$$2^{10} = \underbrace{2 \cdot \quad \ldots \quad 2}_{10} = 2^{2\cdot 5} = (2^2)^5 = (2^5)^2$$

$$2^2 = 2 \cdot 2$$

$$(2^2)^5 = (2\cdot 2)\cdot(2\cdot 2)\cdot(2\cdot 2)\cdot(2\cdot 2)\cdot(2\cdot 2)$$

$a^0 = 1 \quad \underline{\forall a > 0}$ anche per $a < 0$

$$a^{3+0} = \underset{\uparrow}{a^3} = \underset{\uparrow}{a^3} \cdot \underline{\underline{a^0}}$$

a^0 è l'elemento neutro della moltiplicazione, cioè 1

Possiamo ricavare due funzioni

a^b

$y(x) = a^x \quad a > 0$

$y(x) = x^b \leftarrow$

$y = x^2 \quad b=2$

$y = x \quad b=1$

$y = 1 \quad b=0$

$y = \sqrt{x} \quad b=\frac{1}{2}$

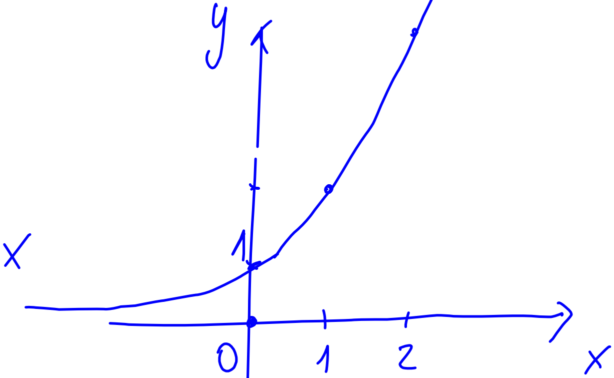
funzione esponenziale

funzione potenza

$a > 1$

$a = 2$

$y = 2^x$

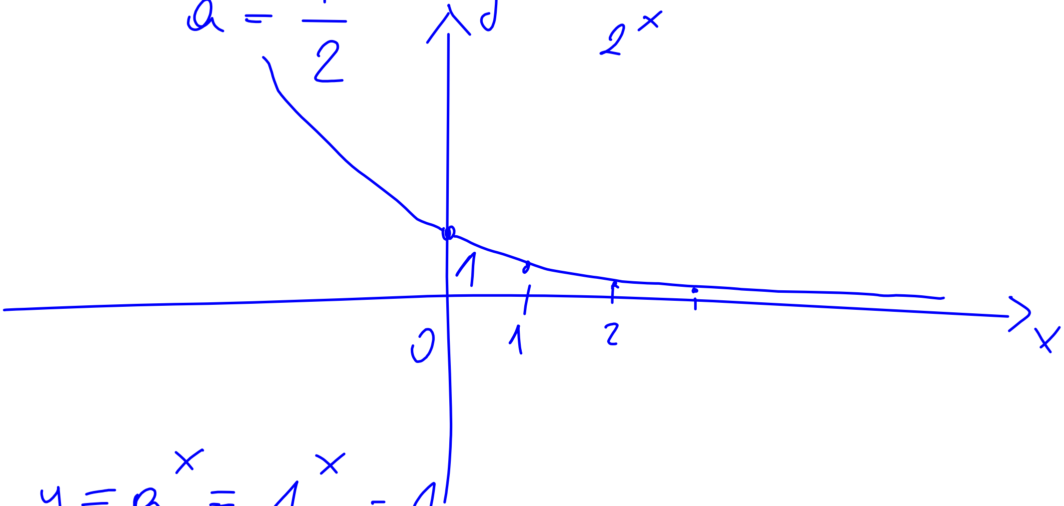


$x \to -x$

$2^x \to 2^{-x} = \frac{1}{2^x}$

$2, 4, 8, 16, 32, 64, 128, 256, 512, 1024 = 2^{10}$

$a < 1$ $\qquad a = \frac{1}{2}$ $\qquad y = \frac{1}{2^x}$

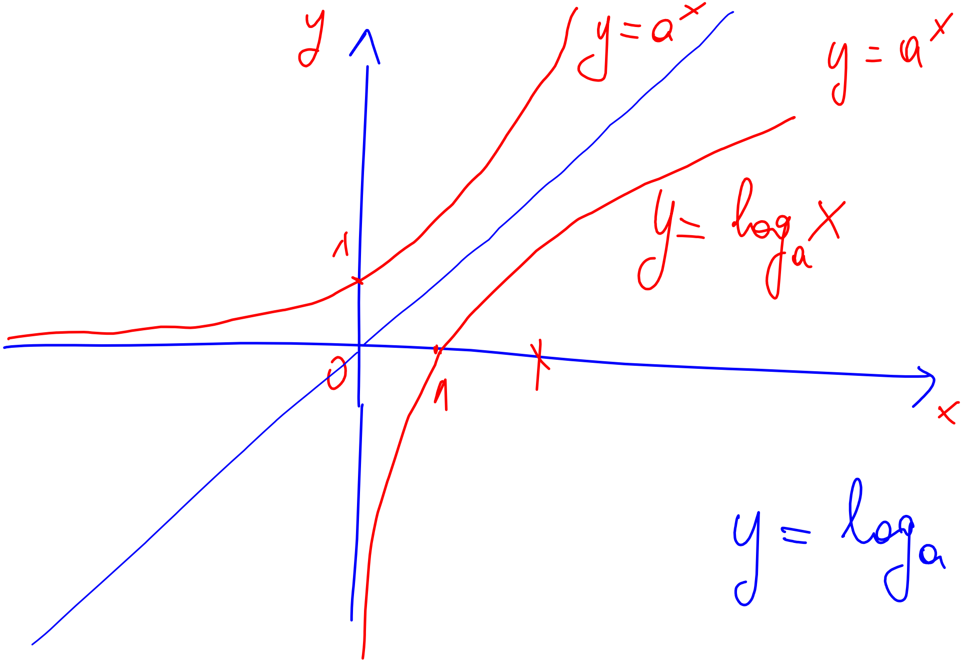


$a = 1$ $\qquad y = a^x = 1^x = 1$

Logaritmo : l'inversa dell'esponenziale

$a > 1$ $\qquad y = a^x \qquad x = \log_a y$

Scambiamo $x \leftrightarrow y$ $\qquad y = \log_a x$



$\mathbb{R} \to \mathbb{R}_+$

codominio

iniettiva e suriettiva, quindi invertibile

$y = \log_a x$ $\quad \mathbb{R}_+ \to \mathbb{R}$

e = numero di Nepero (o di Eulero)

$e = 2.718\ldots$

$y = e^x \qquad x = \ln y = \log_e y$ logaritmo naturale

$y = f(x) \qquad x = f^{-1}(y) \xrightarrow{\text{scambio } x \leftrightarrow y} y = f^{-1}(x)$

$y = f(f^{-1}(x)) = x = f^{-1}(f(x))$

$y = x$ funzione identica

Esempio $\qquad y = x^2 = f(x) \qquad x > 0$

$y = \sqrt{x} = f^{-1}(x)$

$y = f(\underbrace{f^{-1}(x)}) = (\sqrt{x})^2 = x$

$y = f^{-1}(f(x)) = \sqrt{f(x)} = \sqrt{x^2} = x$

$$y = a^x = f(x)$$

$$x = \log_a y \qquad x \leftrightarrow y \qquad\qquad y = \log_a x = f^{-1}(x)$$

$$y = f\left(f^{-1}(x)\right) = a^{f^{-1}(x)} = a^{\log_a x} = x$$

$$\boxed{e^{\ln x} = x} \quad (x > 0)$$

$$y = f^{-1}\left(f(x)\right) = \log_a f(x) = \log_a a^x$$

$$\ln e^x = x$$

a

$$\boxed{a^{b+c} = a^{b} \cdot a^{c}} \qquad \underline{a^{b \cdot c} = (a^{b})^{c} = (a^{c})^{b}}$$

$$\underbrace{\ln_a (bc)}_{s} = \underbrace{\ln_a b}_{t} + \underbrace{\ln_a c}_{u}$$

$$s = t + u$$

$$\begin{cases} s = \ln_a (bc) \\ t = \ln_a b \\ u = \ln_a c \end{cases} \qquad \begin{array}{l} a^{s} = bc \\ a^{t} = b \\ a^{u} = c \end{array}$$

$$b \cdot c = a^{s} = b \cdot c = \boxed{a^{t} \cdot a^{u} = a^{t+u}}$$

Cambio di base

$$y = \underbrace{\log_a x}_{s} = \underbrace{\left(\log_a b\right)}_{t} \cdot \underbrace{\left(\log_b x\right)}_{u} \qquad s = t \cdot u$$

$s = \log_a x \qquad a^s = x$

$t = \log_a b \qquad a^t = b$

$u = \log_b x \qquad b^u = x$

$$a^s = x = b^u = \boxed{\left(a^t\right)^u = a^{t \cdot u}}$$

1 milione 10^6 Mega

1 miliardo 10^9 Giga

mille 10^3 Kilo

$\log_{10}(1 \text{ milione}) = 6$

esponente x a cui elevare 10 per dare 1 milione

$\log_{10}(1 \text{ miliardo}) = 9$

$a = 10$ $b = e$

$y = \log_{10} x = (\log_{10} e)$ $\underline{\underline{\ln x}}$

0.434294 ...

$$y = a^{x} = (a)^{x} = (e^{\ln a})^{x} = e^{\ln a \cdot x}$$

$$a = e^{\ln a}$$

$$e^{2x} \quad e^{3x} \dots$$

$$e^{-x} \dots$$

$$\log_a 1 = 0 \qquad (a>0)$$

$$\log_a a = 1 \qquad a>0$$

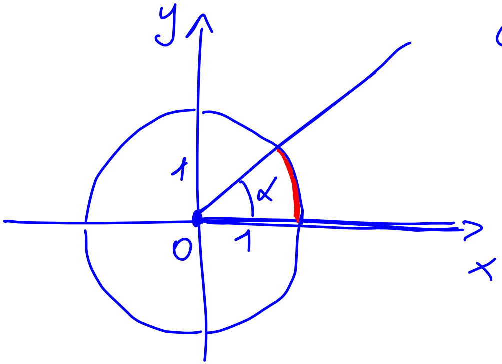
$$a^{\log_a b} = b$$

$$\log_a \frac{b}{c} = \log_a b - \log_a c$$

$$\underbrace{\log_a\left(\frac{b}{c}\right) + \log_a(c)}_{} = \log_a b = \log_a \frac{b}{c}\, c$$

Funzioni trigonometriche

Angoli

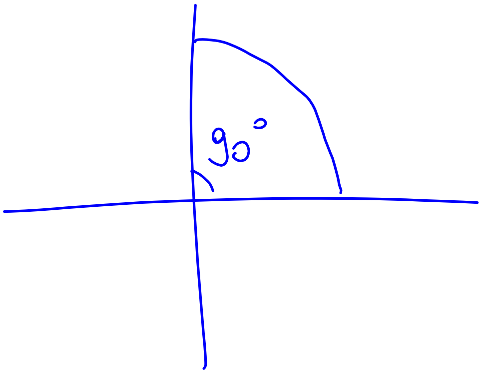


circonferenza di raggio 1

la misura di α in radianti è la lunghezza dell'arco sotteso da α

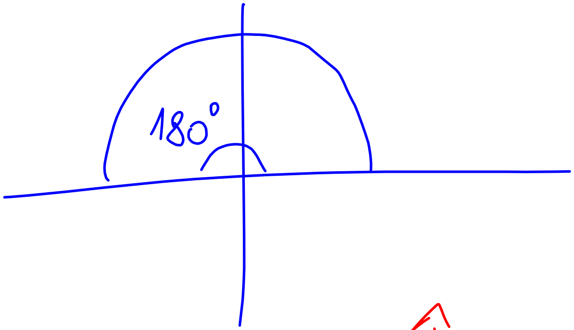
Angolo giro (360°) = la lunghezza della circonferenza di raggio $1 = 2\pi$

circ. di raggio $R = 2\pi R$

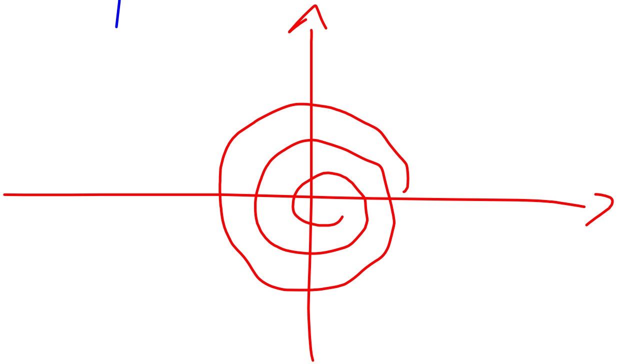


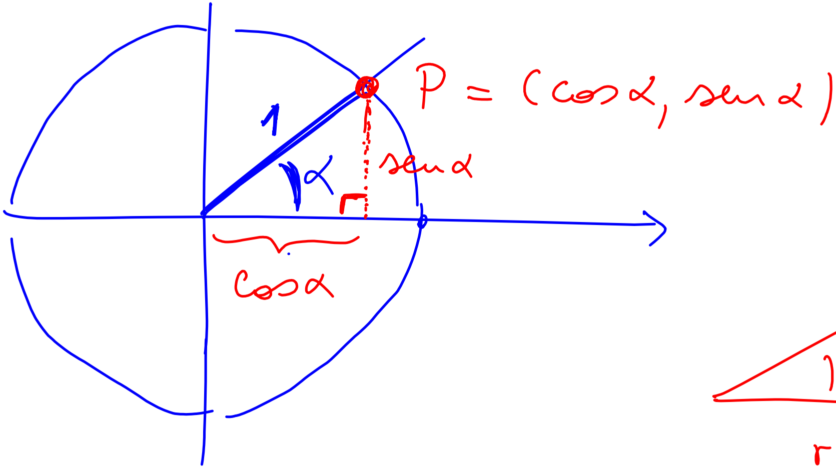
$$\frac{\cancel{90^\circ}}{\cancel{360^\circ}\,\cancel{4}_2}\,\cancel{2}\pi = \alpha = \frac{\pi}{2}$$

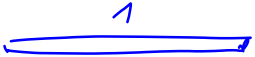
$$90^\circ : 360^\circ = \alpha : 2\pi$$

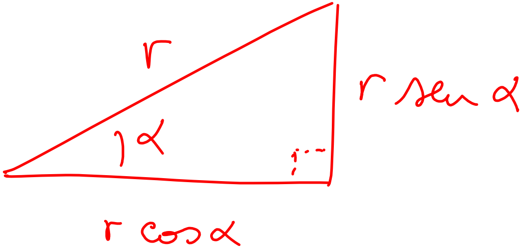


$$\frac{\pi}{2} \cdot 2 = \pi$$









Teorema di Pitagora :

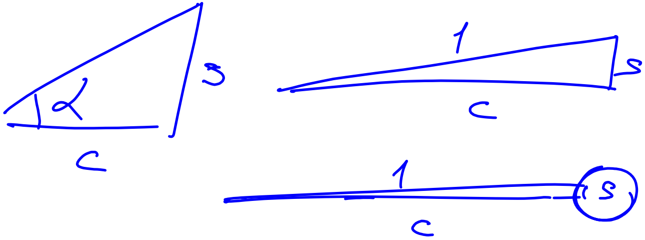
$$\boxed{(\cos\alpha)^2 + (\operatorname{sen}\alpha)^2 = 1}$$

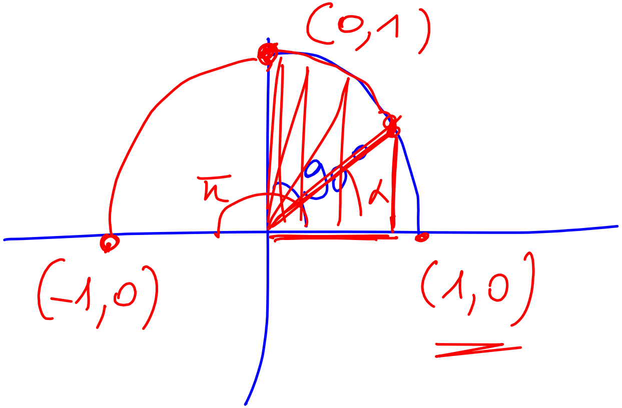
$\operatorname{sen} 0 = 0$

$\underline{\cos 0 = 1}$

$\cos^2 \alpha$

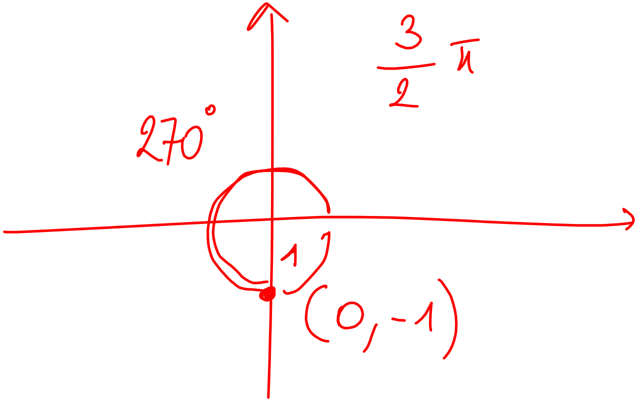
(NON è $\cos(\alpha^2)$)



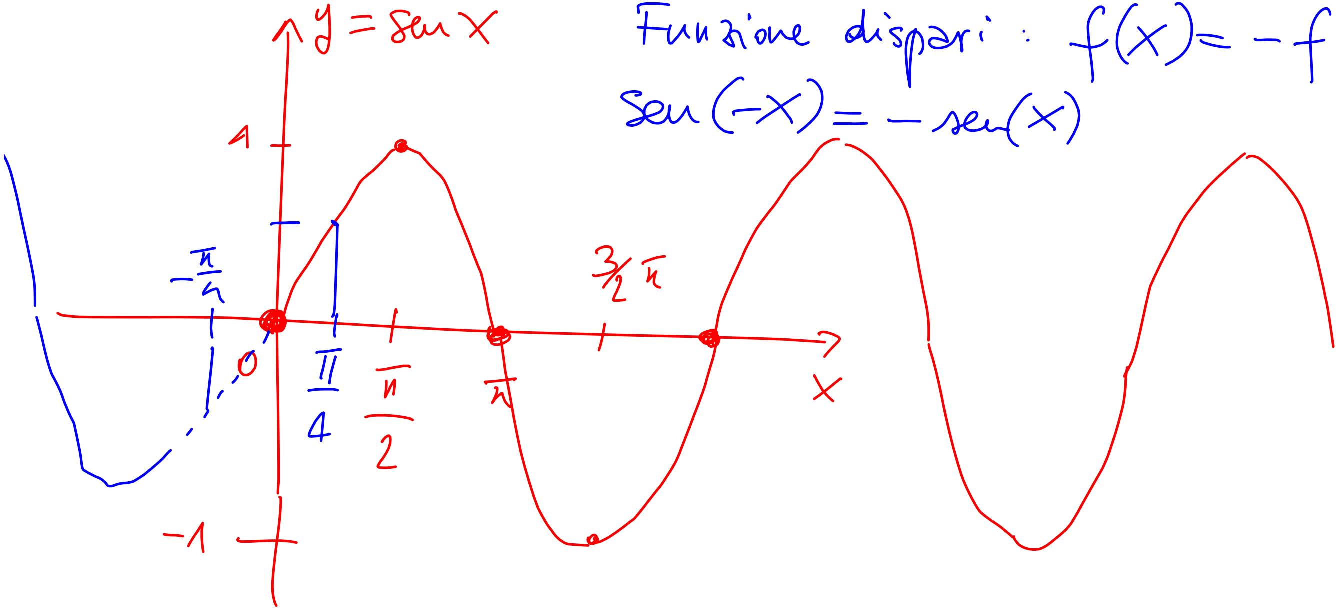


$$\begin{cases} \text{sen}\,\frac{\pi}{2} = 1 \\ \cos\frac{\pi}{2} = 0 \end{cases}$$

$$\begin{cases} \cos\pi = -1 \\ \text{sen}\,\pi = 0 \end{cases}$$



$$\begin{cases} \cos\left(\frac{3}{2}\pi\right) = 0 \\ \text{sen}\left(\frac{3}{2}\pi\right) = 1 \end{cases}$$



Funzione dispari : $f(x) = -f(-x)$

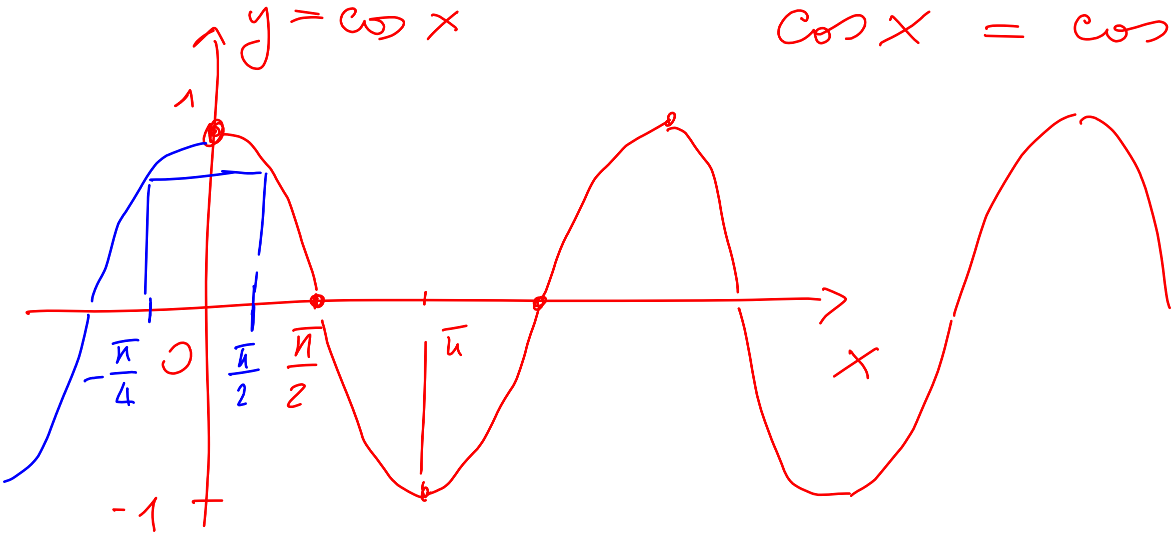
$sen(-x) = -sen(x)$

funzione periodica di periodo a =

funzione $f(x)$ tale $f(x) = f(x+a)\ \forall x$

$sen(x) = sen(x+2\pi)$

$\cos x = \cos(x+2\pi)$



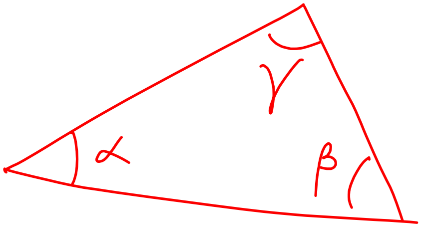
sen(2x)   sen(3x)   sen(ax)

a = numero

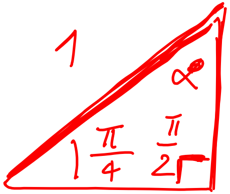
Il coseno è una funzione pari (f(x)=f(-x))

$\cos(-x) = \cos(x)$

α	$\cos\alpha$	$\operatorname{sen}\alpha$
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$



$$\alpha+\beta+\gamma=\pi$$



$$\alpha=\pi-\frac{\pi}{2}-\frac{\pi}{4}=\frac{4-2-1}{4}\pi$$

$$=\frac{\pi}{4}$$

Triangolo isoscele rettangolo

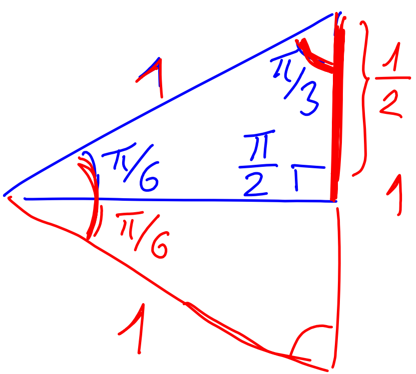
$$\cos\frac{\pi}{4}=\operatorname{sen}\frac{\pi}{4}\equiv x$$

$$\left(\cos\frac{\pi}{4}\right)^2+\left(\operatorname{sen}\frac{\pi}{4}\right)^2=1$$

$$x^2+x^2=1=2x^2$$

$$x=\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2}=\frac{\sqrt{\cancel{2}}}{\sqrt{2}\cdot\sqrt{\cancel{2}}}$$

Triangolo equilatero



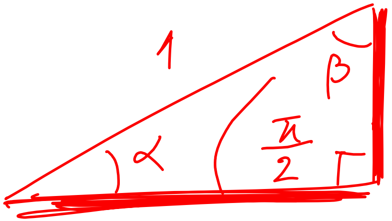
$$\left(\frac{1}{2}\right)^2$$
$$\shortparallel$$

$$\text{sen}\,\frac{\pi}{6} = \frac{1}{2}$$

$$\left(\cos\frac{\pi}{6}\right)^2 + \left(\text{sen}\,\frac{\pi}{6}\right)^2 = 1$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\left(\cos\frac{\pi}{6}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$$



$$\beta = \pi - \alpha - \frac{\pi}{2} = \frac{\pi}{2} - \alpha$$

$$\boxed{\cos\left(\frac{\pi}{2} - \alpha\right) = \operatorname{sen} \alpha}$$

$$\cos \alpha = \operatorname{sen}\left(\frac{\pi}{2} - \alpha\right)$$

$$-1 \leq \operatorname{sen} \alpha \leq 1$$

$$-1 \leq \cos \alpha \leq 1$$

un cateto non può essere più lungo dell' ipotenusa

Identità

$$\operatorname{sen}(\alpha+\beta)=\operatorname{sen}\alpha\cos\beta+\operatorname{sen}\beta\cos\alpha$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\operatorname{sen}\alpha\operatorname{sen}\beta$$

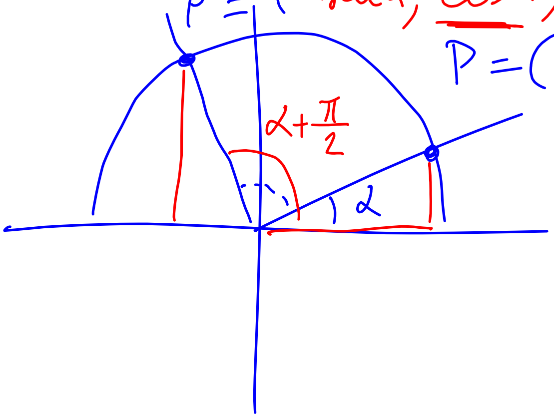
$$\operatorname{sen}\left(\alpha+\frac{\pi}{2}\right)=\cancel{\operatorname{sen}\alpha\underset{\overset{\|}{0}}{\cos\frac{\pi}{2}}}+\underset{\overset{\|}{1}}{\operatorname{sen}\frac{\pi}{2}}\cos\alpha$$

$$=\cos\alpha$$

$$\beta=\frac{\pi}{2}$$

$P'=(-\operatorname{sen}\alpha,\ \underline{\cos\alpha})$

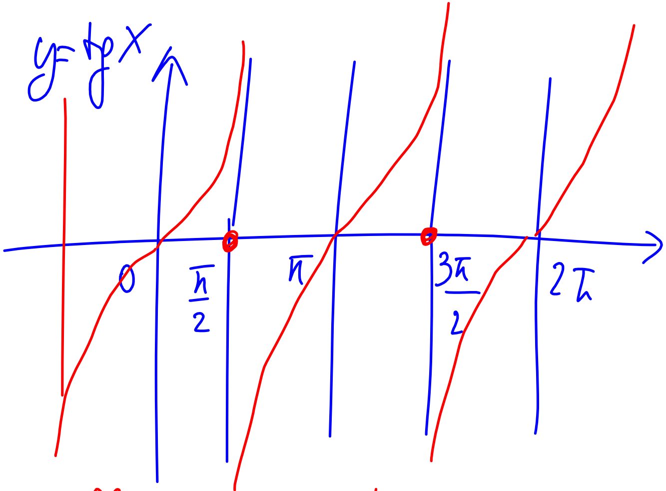
$P=(\cos\alpha,\ \operatorname{sen}\alpha)$

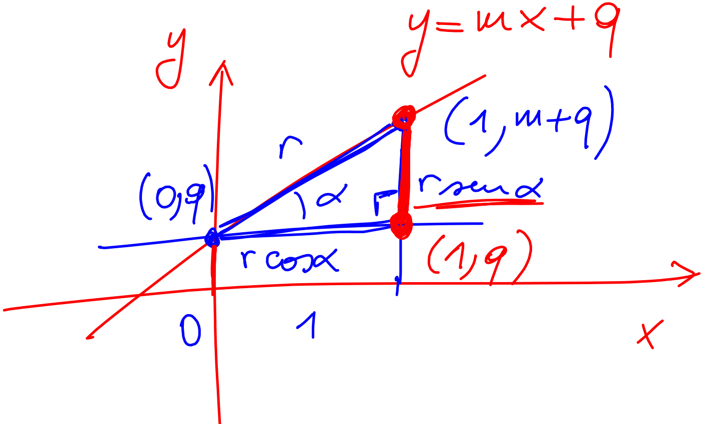


$$\cos\left(\alpha+\frac{\pi}{2}\right)=\cancel{\cos\frac{\pi}{2}}\cos\alpha-\operatorname{sen}\frac{\pi}{2}\operatorname{sen}\alpha=-\operatorname{sen}\alpha$$

Tangente e cotangente

$$\operatorname{tg} x = \frac{\operatorname{sen} x}{\cos x} \qquad \operatorname{cotg} x = \frac{\cos x}{\operatorname{sen} x}$$



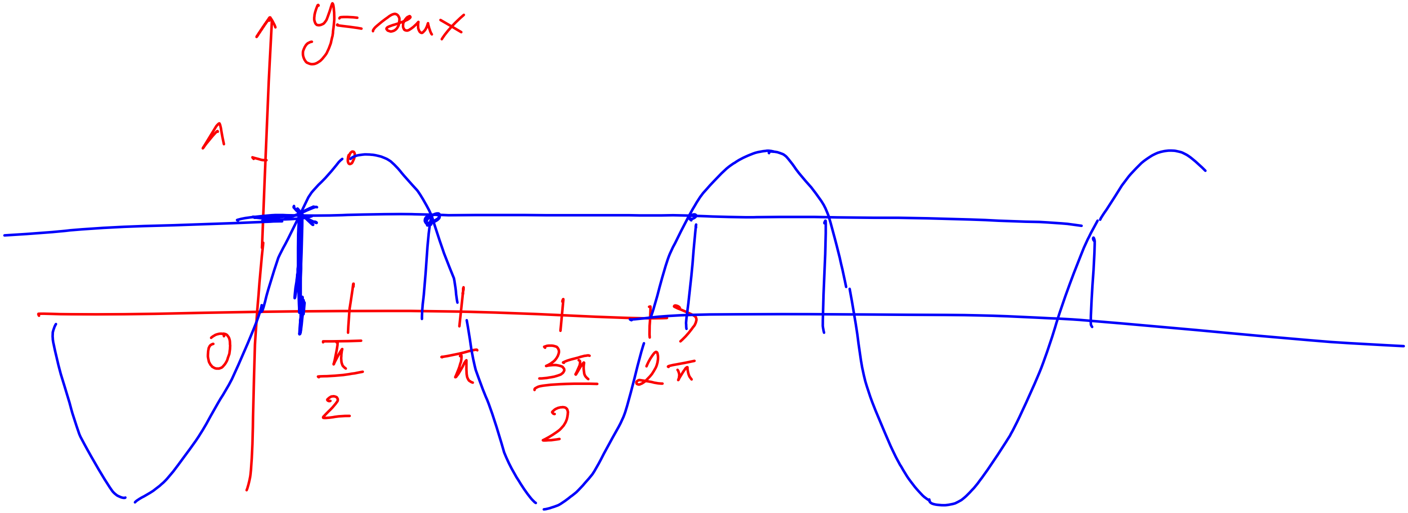


Il coefficiente angolare di una retta è la tangente dell'angolo formato da quella retta e l'orizzontale

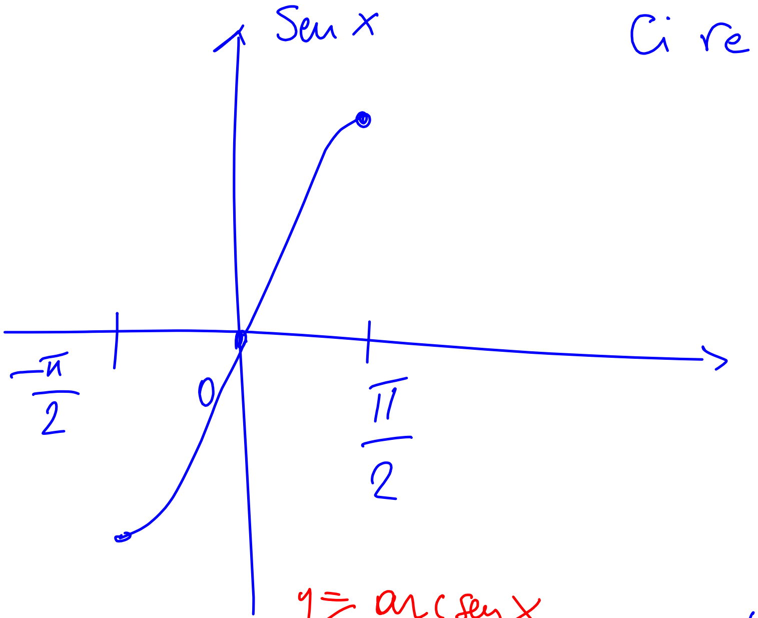
$m = \operatorname{tg} \alpha$ $\qquad r \cos \alpha = 1 \qquad r \operatorname{sen} \alpha = m$

$$\frac{\cancel{r} \operatorname{sen} \alpha}{\cancel{r} \cos \alpha} = \frac{m}{1} \qquad m = \operatorname{tg} \alpha$$

Funzioni inverse



non è invertibile, perché a ogni ordinata corrispondono più (in realtà infinite) ascisse



Ci restringiamo al dominio

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

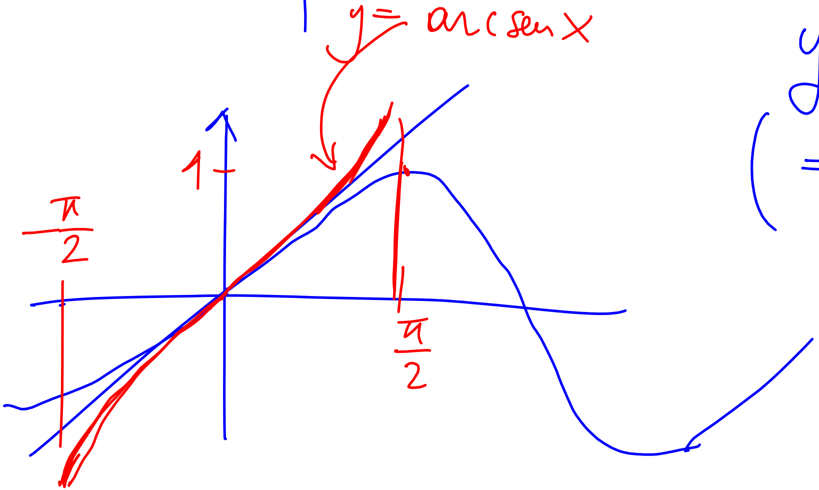
Qui è invertibile

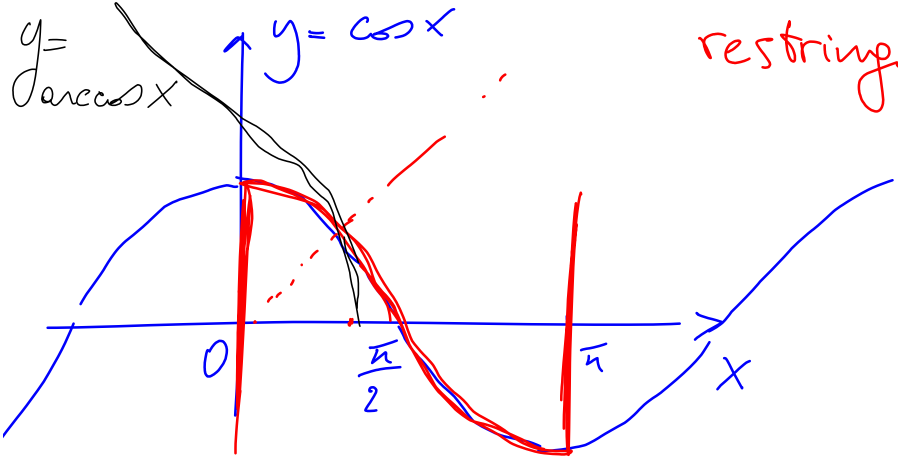
L'inversa si chiama

arcoseno

$$y = \text{arcsen}(x)$$

$$\left(\Longrightarrow \text{sen}\, y = x \right)$$





restringo il dominio a

$[0, \pi]$

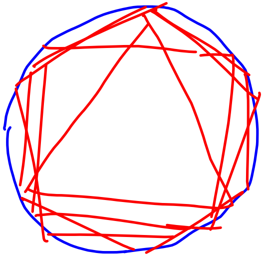
qui esiste

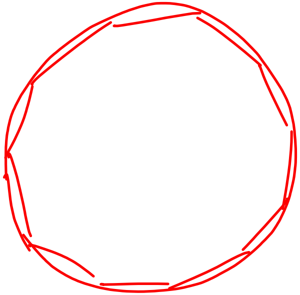
l'inversa

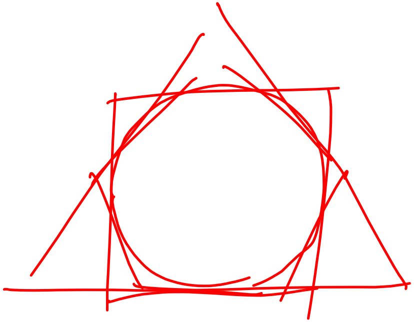
$y = \arccos(x)$

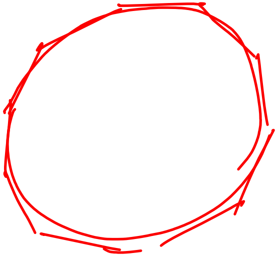
$[\Rightarrow \cos y = x]$

I LIMITI









circonf. < perimetri di tutti i poligoni
circoscritti

circonf. > perimetri di tutti i poligoni
inscritti

(circonferenza di raggio 1)

Archimede : $\exists$ 1 solo numero (2π)
colle proprietà richieste

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$a_n = \frac{1}{n} \qquad n = 1, 2, 3, \ldots$$

successione

$$a_n = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$$

$$\ldots \frac{1}{10^6} \ldots \frac{1}{10^9}$$

Definizione di limite di una successione

$$\lim_{n\to\infty} a_n = \ell \qquad \text{vuol dire}$$

$$\forall \varepsilon > 0 \quad \exists N : \quad \forall n > N \quad |\ell - a_n| < \varepsilon$$

$$\lim_{n\to\infty} \frac{1}{n} = 0 \qquad a_n = \frac{1}{n} \qquad \ell = 0$$

$$\underline{\underline{\forall \varepsilon > 0}} \quad \exists N : \quad \forall n > N \quad \frac{1}{n} < \varepsilon$$

Suggerimento : $N = \frac{1}{\varepsilon}$

$\forall \varepsilon$ scelgo $N = \frac{1}{\varepsilon}$, allora verifico che

$\boxed{\forall n > N = \frac{1}{\varepsilon}}$ vale $\boxed{\frac{1}{n} < \varepsilon}$

$n > \frac{1}{\varepsilon}$ $\quad \varepsilon n > 1$ $\quad \boxed{\varepsilon > \frac{1}{n}}$

$\left[a < b \Rightarrow \lambda a < \lambda b \quad \forall \lambda > 0 \right]$

Limite di una funzione : $f(x)$

$\lim_{x \to x_0} f(x) = \ell$ vuol dire :

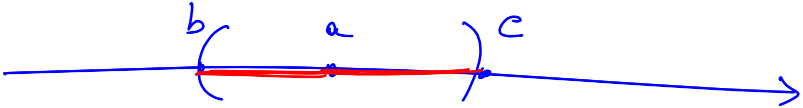
$\forall \varepsilon > 0 \quad \exists \delta > 0 \; / \; \forall x : |x - x_0| < \delta \quad x \neq x_0$

vale $|f(x) - \ell| < \varepsilon$

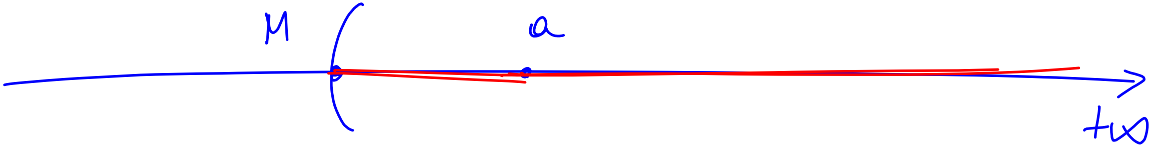
Si dice intorno di $a \in \mathbb{R}$ un qualunque intervallo aperto che contiene a

Se $a \neq \pm\infty$ (b, c) è un intorno

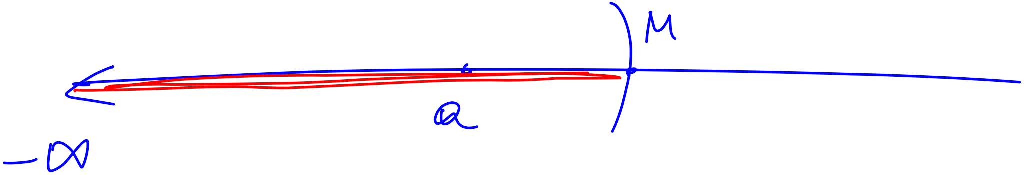
$$b < a < c$$



Se $a = +\infty$ un intorno è qualunque intervallo (M, ∞) con $M < a$



Se $a = -\infty$ $(-\infty, M)$ con $M > a$



Diciamo che una funzione $f(x)$ definita in un intorno di x_0 (ma non necessariamente in x_0) tende al limite ℓ (che potrebbe essere ∞) per x che tende a x_0 (che potrebbe essere ∞) e si scrive $\lim_{x \to x_0} f(x) = \ell$,

se $\forall\, I_\ell$ intorno di ℓ

$\exists\, I_{x_0}$ intorno di x_0 tale che

$$x \in I_{x_0} \quad x \neq x_0 \implies f(x) \in I_\ell$$

Se $f(x)$ è continua in x_0 allora

$$\lim_{x \to x_0} f(x) = f(x_0)$$

$$y = f(x) = x^2$$

$$\lim_{x \to 2} f(x) = 4$$

$$\lim_{x \to 3} f(x) = 9$$

$\lim_{x \to \infty} f(x) = \ell$ vuol dire

$$\forall \varepsilon > 0 \quad \exists M \;/\; \forall x > M \quad |f(x) - \ell| < \varepsilon$$

$\lim_{x \to -\infty} f(x) = \ell \iff$ (se e solo se)

$$\forall \varepsilon > 0 \quad \exists M \;/\; \forall x < M \quad |f(x) - \ell| < \varepsilon$$

$(M < 0)$

$$\lim_{x \to \infty} \frac{x \boxed{+1}}{x \boxed{-1}} = 1$$

$+1$ è trascurabile rispetto a x per x grande

$$\forall \varepsilon > 0 \quad \exists(?) \quad M > 0 : \quad \boxed{\forall x > M}$$

$$\left| \frac{x+1}{x-1} - 1 \right| < \varepsilon$$

$$\downarrow$$

$$\left| \frac{\cancel{x}+1-\cancel{x}+1}{x-1} \right| = \boxed{\frac{2}{x-1} < \varepsilon}$$

(x è grande, quindi posso supporre $x>1$)

$$2 < \varepsilon(x-1) \qquad \frac{2}{\varepsilon} < x-1 \qquad \underbrace{1+\frac{2}{\varepsilon}}_{M} < x$$

$$M < x$$

$$\lim_{x \to \infty} \frac{3x+2}{2x+1} = \frac{3}{2}$$

$$\lim_{x \to \infty} \frac{\boxed{7x^2} + 8x \cancel{+2}}{\boxed{3x^2} + 4x \cancel{+1}} = \lim_{x \to \infty} \frac{7x^{\cancel{2}} + 8\cancel{x}}{3x^{\cancel{2}} + 4\cancel{x}} =$$

$$= \lim_{x \to \infty} \frac{7x+8}{3x+4} = \frac{7}{3}$$

$$\lim_{x \to \infty} \frac{\boxed{8x^4} + 7x^3 + 2x^2 + 1}{\boxed{7x^4} + 8x^3 + 2x^2 + 3} = \frac{8}{7}$$

$$\lim_{x\to\infty} \frac{\boxed{7x^4}+1}{\boxed{2x^3}+1} = \lim_{x\to\infty} \frac{7x^{\cancel{4}}}{2\cancel{x^3}} = \lim_{x\to\infty} \frac{7x}{2} = +\infty$$

$$\lim_{x\to\infty} \frac{\boxed{2x^3}+4x^2+1}{\boxed{7x^4}+8x^3+2x^2+4} = \lim_{x\to\infty} \frac{2\cancel{x^3}}{7x^{\cancel{4}}}$$

$$= \lim_{x\to\infty} \frac{2}{7x} = 0$$

Proprietà

$$\left[\lim_{x\to x_0} \left(f(x) + g(x)\right) = \underbrace{\lim_{x\to x_0} f(x)}_{} + \underbrace{\lim_{x\to x_0} g(x)}_{}\right.$$

se entrambi esistono e sono $\neq \pm\infty$]

$$\lim_{x\to x_0} \left(x - x\right) = 0 = \underbrace{\boxed{\lim_{x\to x_0} x}}_{= x_0} - \underbrace{\boxed{\lim_{x\to x_0} x}}_{= x_0}$$

$(x_0 \neq \pm\infty)$

$$\lim_{x\to\infty} \underbrace{(x-x)} = 0 \neq \boxed{\lim_{x\to\infty} x} - \lim_{x\to\infty} x$$

ha senso e fa ∞

$\infty - \infty$ NON ha senso

$$\lim_{x\to\infty} \left[(1+x) - x\right] = 1 \neq \underbrace{\lim_{x\to\infty} (1+x)}_{\infty} - \underbrace{\lim_{x\to\infty} x}_{\infty}$$

$$\lim_{x\to x_0} f(x)\cdot g(x) = \underbrace{\lim_{x\to x_0} f(x)} \cdot \underbrace{\lim_{x\to x_0} g(x)}$$

se entrambi esistono e sono finiti

oppure uno è finito $\neq 0$ e l'altro è $\pm\infty$

NON HANNO SENSO

$\boxed{\frac{a}{0}}$ $\frac{\infty}{0}$

a qualunque

$\boxed{\infty - \infty}$

$\boxed{0 \cdot \infty}$ $\boxed{\frac{\infty}{\infty}}$

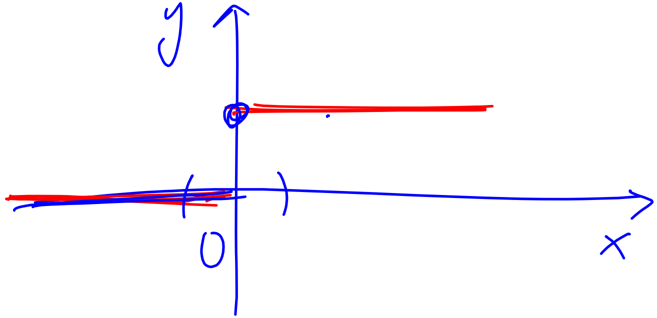
Queste cose vanno evitate

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$$

$f = 1$

$g = \frac{x}{\operatorname{sen} x}$

se non capita una situazione come quelle elencate sopra



$$f(x) = \begin{cases} 1 & \text{per } x > 0 \\ 0 & \text{per } x < 0 \end{cases}$$

$$\lim_{x \to 0} f(x)$$

x solo positivi : il lim è 1

x solo negativi : il lim è 0

$$I_{x_0} = (x_0 - \delta, x_0 + \delta)$$

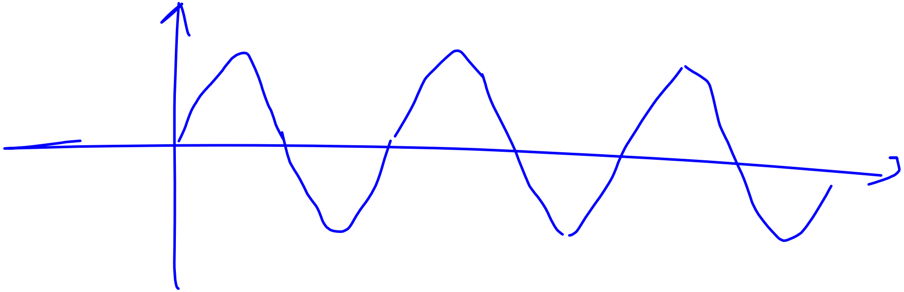
limite destro : I_{x_0} è sostituito da un intorno destro $(x_0, x_0 + \delta)$

limite sinistro : I_{x_0} è sostituito da un intorno sinistro $(x_0 - \delta, x_0)$

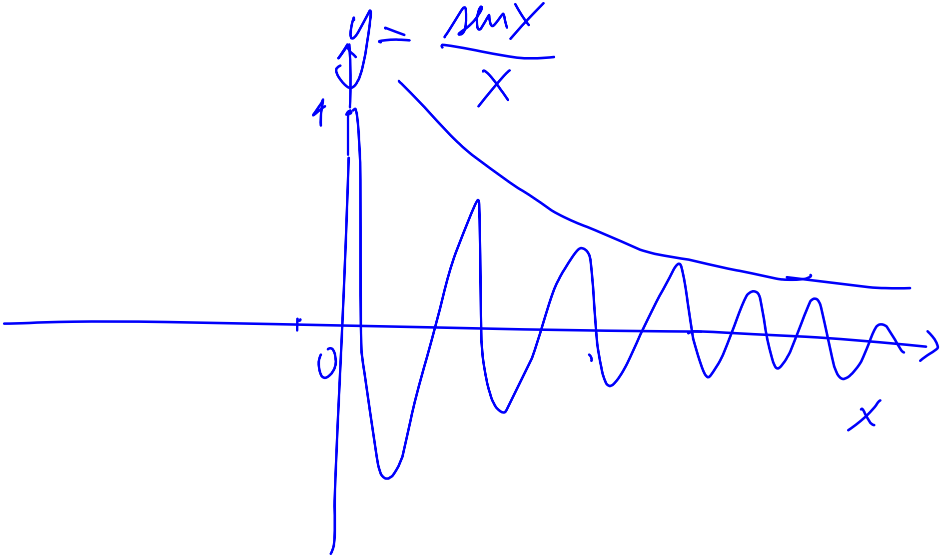
Il salto di una funzione discontinua è
la differenza tra il limite destro e il
limite sinistro

Non sempre il limite esiste

$$\lim_{x \to \infty} \operatorname{sen}(x) = \,?\quad \text{non esiste}$$



$$\lim_{x \to \infty} \frac{\operatorname{sen}(x)}{x} = 0$$



$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$

$$\frac{0}{0}$$

Teorema del confronto (o dei carabinieri)

Se $f(x) \leq g(x) \leq h(x)$ $\forall x \in D$

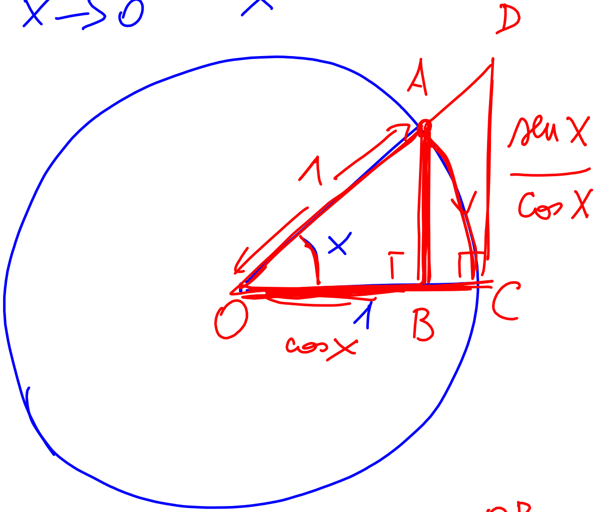
e $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \ell$ $x_0 \in D$

allora anche $\lim_{x \to x_0} g(x) = \ell$

$\pi = 3.14$

$$\lim_{x \to 0} \frac{\text{sen}\,x}{x}$$

$\overline{AO} = 1$



$\forall x > 0$

$$\boxed{\begin{array}{l} \text{area } \widehat{AOB} \le \\ \text{area } AOC \le \\ \text{area } \widehat{DOC} \end{array}}$$

$$\begin{array}{cc} OB & AB \end{array}$$

$$\text{area } \widehat{AOB} = \frac{\cos x \cdot \text{sen}\,x}{2}$$

$\boxed{OB} = \cos x$

$OC = 1$

$$\text{area } AOC = \frac{x}{2\cancel{\pi}}\cancel{\pi} = \frac{x}{2}$$

$$\text{area } \widehat{DOC} = \frac{\cancel{\cos x} \cdot \text{sen}\,x}{2} \frac{1}{\cancel{\cos x}} \frac{1}{\cos x} = \frac{\text{sen}\,x}{2\cos x}$$

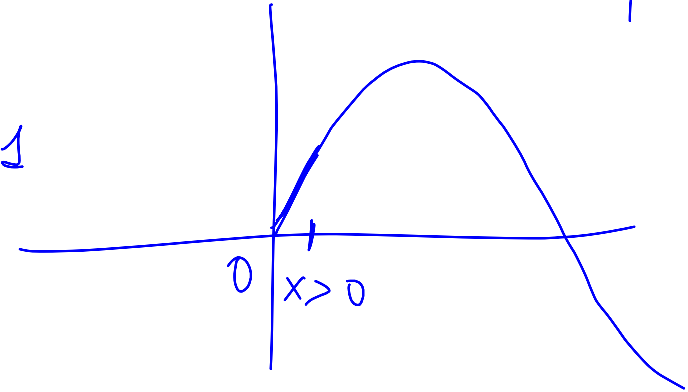
$$\frac{\operatorname{sen}x \cdot \cos x}{\cancel{2}} \le \frac{x}{\cancel{2}} \le \frac{\operatorname{sen}x}{\cancel{2}\cos x}$$

divido per sen x
(x ≠ 0 e
x piccolo > 0
quindi sen x ≠ 0)
sen x > 0

$$\cos x \le \frac{x}{\operatorname{sen}x} \le \frac{1}{\cos x}$$

$x \to 0 \; : \; \cos x \to 1$

$x \to 0 \; : \; \dfrac{1}{\cos x} \to 1$



$$\lim_{x \to 0} \frac{x}{\operatorname{sen}x} = 1$$

$$\lim_{x \to 0} \frac{\operatorname{sen}x}{x} = \lim_{x \to 0} \frac{1}{\frac{x}{\operatorname{sen}x}} = 1$$

lunghezza circonferenza di raggio r : $2\pi r$

area : πr^2

area sfera di raggio r : $4\pi r^2$

Volume della sfera : $\frac{4\pi}{3} r^3$

$$\lim_{x\to 0}\left(\frac{\operatorname{sen}x}{x}\right)^2 = \lim_{x\to 0}\frac{\operatorname{sen}x}{x}\cdot\lim_{x\to 0}\frac{\operatorname{sen}x}{x}$$

$2x=y \quad y=\frac{x}{2}$

$$\lim_{x\to 0}\frac{\operatorname{sen}(2x)}{2x}\cdot 2 = 2\cdot\lim_{x\to 0}\frac{\operatorname{sen}(2x)}{2x} =$$

$$= 2\lim_{y\to 0}\frac{\operatorname{sen}y}{y} = 2\lim_{x\to 0}\frac{\operatorname{sen}x}{x}$$

limite della composizione di funzioni

Se $\lim_{x \to x_0} g(x) = y_0$ e

f è continua in y_0 allora

$$\lim_{x \to x_0} f(g(x)) = f(y_0)$$

Potete fare un cambio di variabili

$$\lim_{x \to 0} \frac{tg(x)}{x} = \lim_{x \to 0} \frac{\operatorname{sen} x}{x \cdot \cos x} = \lim_{x \to 0} \frac{\operatorname{sen} x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} = 1$$

$$\lim_{x\to 0} \frac{tg(5x)}{5x}5 = \lim_{5x\to 0} \frac{tg(5x)}{5x}\cdot 5 = 5$$

$5x \dots x$

Altri limiti notevoli:

$$\lim_{x\to 0} e^x = 1$$

$$\boxed{\lim_{x\to 0} \frac{e^x-1}{x} = 1} \longrightarrow \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

$$\boxed{1+x=e^t} \qquad x=e^t-1 \qquad \ln(1+x)=\ln e^t = t$$

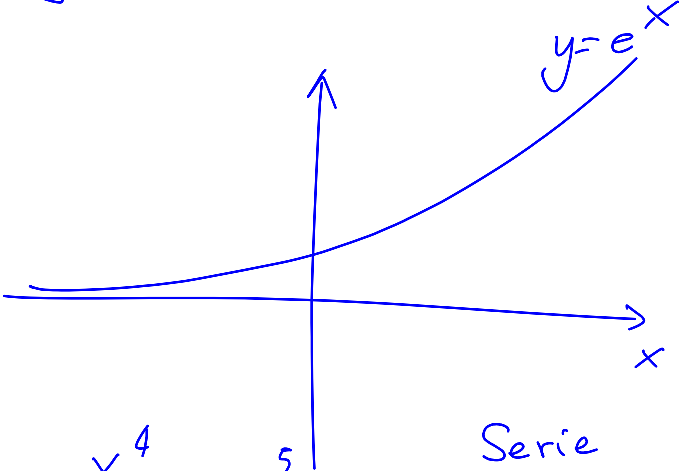
$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{t\to 0} \frac{t}{e^t-1} = 1$$

$$\lim_{x\to\infty} \frac{\boxed{e^x}}{x^n} = \infty \quad \leftarrow$$

$$\frac{\infty}{\infty}$$

$n = \quad 1, 2, 3, \ldots$

$x^n = \quad x, x^2, x^3, x^4$



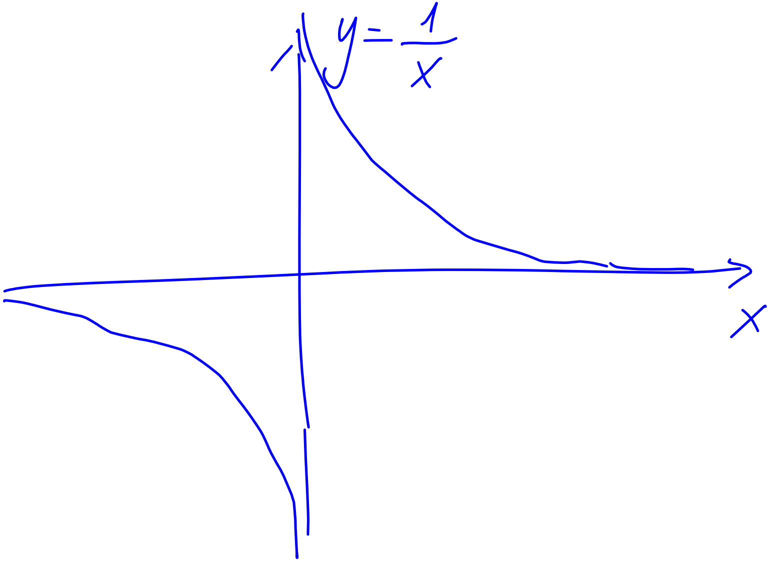
Serie

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

$$e = e^0 = 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0 \quad \leftarrow \qquad n = 1, 2, \ldots$$

$y = \frac{1}{x}$



$$\lim_{x\to 0} \frac{1}{x} = ?$$

NON ESISTE

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$

limite destro

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

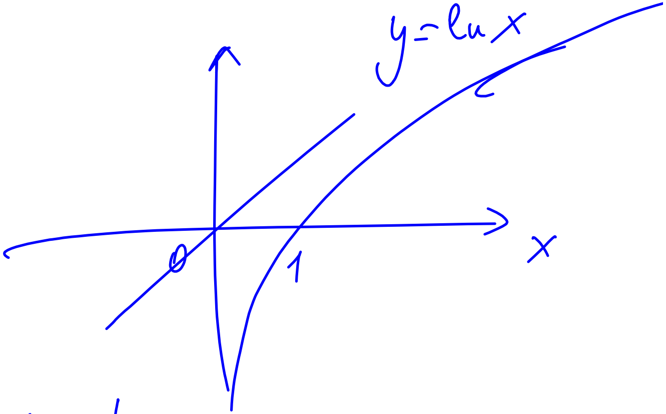
limite sinistro

$\frac{x}{3} = t$

$$\lim_{x\to \infty} \frac{e^{\frac{x}{3}}}{x^{10}} = \lim_{x\to\infty} \frac{e^{\left(\frac{x}{3}\right)}}{\left(\frac{x}{3}\right)^{10} \cdot 3^{10}}$$

$$= \lim_{t\to\infty} \frac{e^t}{t^{10}} \cdot \frac{1}{3^{10}} = +\infty$$

La crescita logaritmica è più lenta di qualunque potenza



$$\lim_{x \to \infty} \frac{x}{\ln x} = \infty$$

$\ln x = t$ se $x \to \infty$

$x = e^t$ anche $t \to \infty$

$$\parallel$$

$$\lim_{t \to \infty} \frac{e^t}{t} = \infty$$

$$\lim_{x \to \infty} \frac{x}{(\ln x)^n} = \infty$$

$n = 1, 2, 3, \ldots$

$$\lim_{x\to -\infty} e^{x} = 0$$

$$\lim_{x\to -\infty} \boxed{x^{n}}\,\underline{e^{x}}$$

$n = 2, 4, 6, \ldots$

$$\lim_{x\to \infty} \sqrt{x+1} + \sqrt{x} = \infty$$

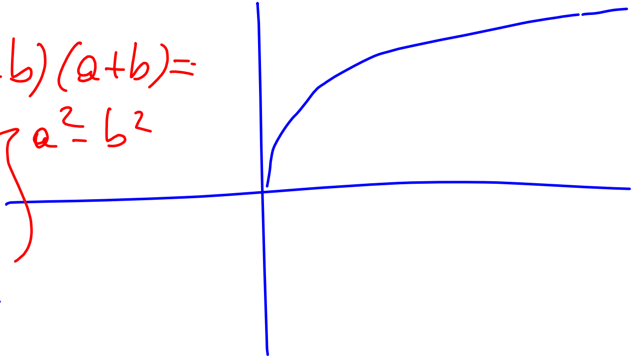
$$\lim_{x\to\infty} \sqrt{x} = \infty$$

$\infty - \infty$

$$\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} =$$

$(a-b)(a+b) = a^{2} - b^{2}$

$$= \lim_{x\to\infty} \frac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{\sqrt{x+1} + \sqrt{x}}$$



$a = \sqrt{x+1} \qquad b = \sqrt{x}$

$$= \lim_{x\to\infty} \frac{\cancel{x}+1 - \cancel{x}}{\sqrt{x+1} + \sqrt{x}} = 0$$

Limite notevole $\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

$$\lim_{x \to 0} \frac{\overset{a-b}{1-\cos x}}{x^2} \frac{\overset{a+b}{1+\cos x}}{1+\cos x} = \lim_{x \to 0} \frac{1-\cos^2 x}{x^2} \frac{1}{1+\cos x} =$$

$$= \lim_{x \to 0} \frac{\operatorname{sen}^2 x}{x^2} \underbrace{\frac{1}{1+\cos x}} = \underbrace{\lim_{x \to 0} \frac{\operatorname{sen}^2 x}{x^2}}_{1} \cdot \underbrace{\lim_{x \to 0} \frac{1}{1+\cos x}}_{1/2} =$$

$$= \frac{1}{2}$$

$\operatorname{sen}^2 x + \cos^2 x = 1$

$$\lim_{x\to 0} \frac{\operatorname{sen}(x^2)}{x^2} = \lim_{t\to 0} \frac{\operatorname{sen} t}{t} = 1$$

$$t = x^2$$

$$\lim_{x\to 0} \frac{\operatorname{sen}(x^2)}{x} = \lim_{x\to 0} \frac{\operatorname{sen}(x^2)}{x^2} \cdot x =$$

$$= \underbrace{\lim_{x\to 0} \frac{\operatorname{sen}(x^2)}{x^2}}_{1} \cdot \underbrace{\lim_{x\to 0} x}_{0} = 0$$

$$\lim_{x\to 0} \frac{\operatorname{sen} x - x}{x} = \lim_{x\to 0} \left(\underset{1}{\frac{\operatorname{sen} x}{x}} - \underset{1}{\frac{\cancel{x}}{\cancel{x}}} \right) = 0$$

$$\lim_{x\to\infty} \underbrace{\frac{x^2-1}{x+2}}_{\substack{\downarrow \\ \infty}} - \underbrace{x}_{\substack{\downarrow \\ \infty}} = \lim_{x\to\infty} \frac{x^2-1-x(x+2)}{x+2} =$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}-1-\cancel{x^2}-2x}{x+2} = \lim_{x\to\infty} \frac{-1-2x}{x+2} = -2$$

Se $f(x)$ è continua in x_0 allora i limiti destro e sinistro per $x \to x_0$ coincidono e sono uguali al limite $\lim_{x\to x_0} f(x) = f(x_0)$

Infinitesimi

Diciamo che $f(x)$ è infinitesima per $x \to x_0$ se $\lim_{x \to x_0} f(x) = 0$

Se $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \lim_{x \to x_0} \frac{f(x)}{g(x)}$

allora si dice che f è un infinitesimo di ordine superiore a g e si scrive $f = o(g)$

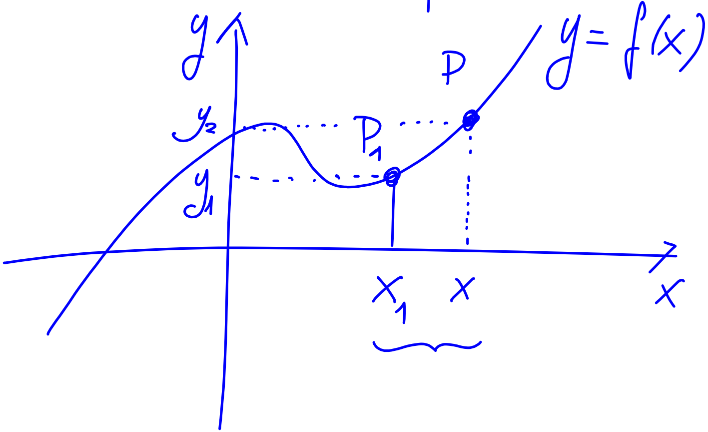
"o piccolo"

Esempio: $f(x) = x^2$ $x_0 = 0$

$g(x) = x$

Derivate

$f(x)$



teniamo x_1 fisso e facciamo tendere x a x_1

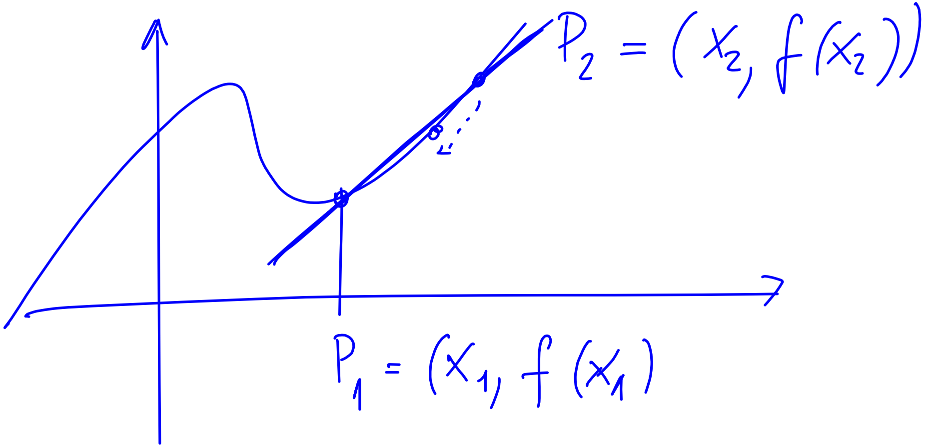
$$\lim_{x \to x_1} \underset{\text{variabile}}{x} - \underset{\text{costante}}{x_1} = 0$$

$$\lim_{x \to x_1} f(x) - f(x_1) = 0$$

Se la funzione $f(x)$ è continua in x_1

$$\lim_{x \to x_1} \underbrace{\frac{f(x)-f(x_1)}{x - x_1}}_{\text{rapporto incrementale}} = ?$$

Se ∃ si chiama derivata di f in x_1 e si indica con



$$y = mx + q$$

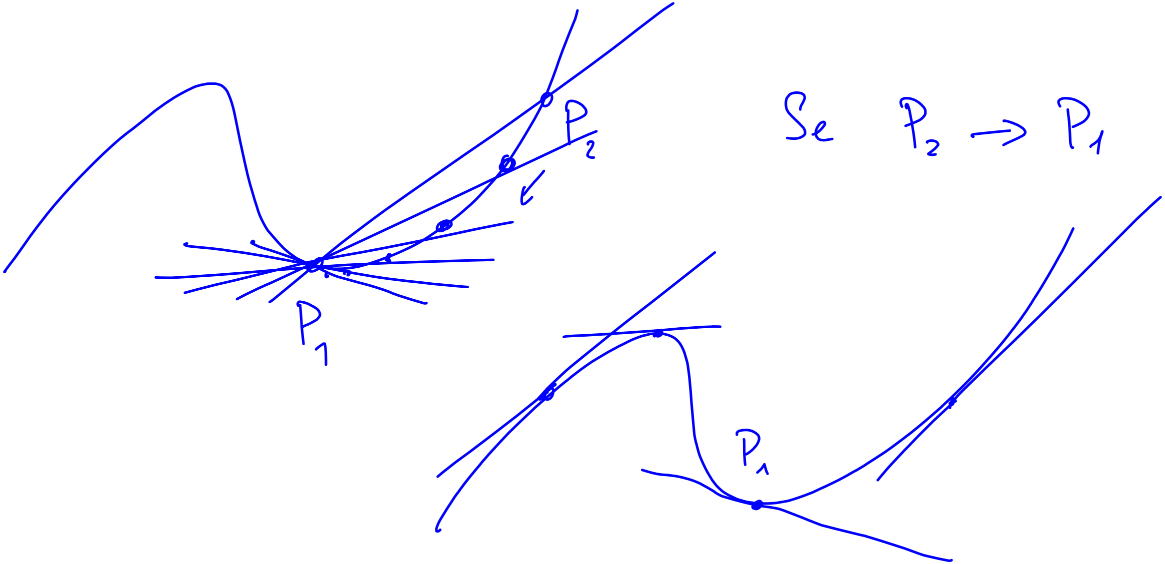
retta che passa per P_1 e P_2

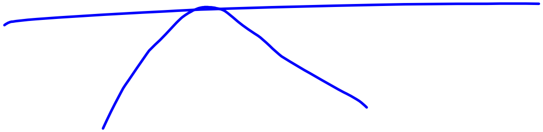
$$y = \underbrace{f(x_1)}_{\text{cost}} + \frac{f(x_2)-f(x_1)}{\underbrace{x_2-x_1}_{\text{cost}}}\,(\underset{\uparrow}{x} - \underbrace{x_{1}}_{\text{cost}})$$

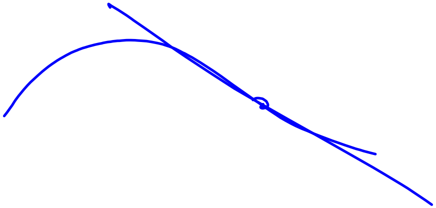
$\longrightarrow$ rapporto incrementale

Infatti: se $x = x_1$ $\quad y = f(x_1)$

se $x = x_2$ $\quad y = f(x_2)$


P_2
P_1
Se $P_2 \to P_1$
P_1



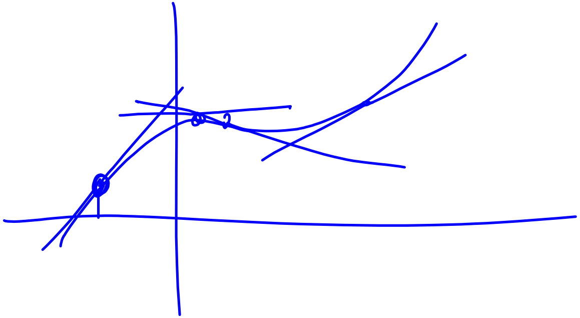


$f'(x_1) \qquad \frac{df}{dx}(x_1)$

di f su di x

de f su de x

Se la derivata di $f(x)$ esiste in x allora la funzione si dice derivabile in x



$f(x)$

$\hookrightarrow f'(x)$

$y = f(x) = 1 \quad \underline{\forall x}$ funzione costante

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(\overbrace{x+\Delta x}^{x_2}) - f(\overset{x_1}{x})}{\underbrace{x+\Delta x}_{x_2} - \underset{x_1}{x}}$$

$(\Delta x \neq 0)$

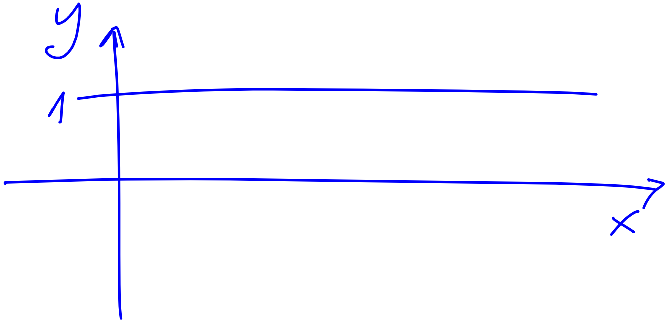
$x + \Delta x = x_2$

$x_1 = 0$

$\Delta x = x_2 - x_1$

$\Delta x \to 0 : x_2 \to x_1$

$$= \lim_{\Delta x \to 0} \frac{1-1}{\Delta x} = 0$$



La derivata di una funzione costante è zero

$f(x)$ $$\frac{df}{dx} = f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

$$\delta = (x+\delta) - x$$

Se $f(x) = c$ costante $$\frac{df}{dx} = 0$$

$$\lim_{\delta \to 0} \frac{c-c}{\delta} = \lim_{\delta \to 0} 0 = 0$$

Se $f(x) = mx + q$ retta

$$\frac{df}{dx} = \lim_{\delta \to 0} \frac{m(\cancel{x}+\delta)+\cancel{q} - (m\cancel{x}+\cancel{q})}{\delta} = \lim_{\delta \to 0} \frac{m\cancel{\delta}}{\cancel{\delta}} = m$$

$$f(x) = x^2 \qquad (a+b)^2 = a^2 + b^2 + 2ab$$

$$\frac{df}{dx} = \lim_{\delta \to 0} \frac{(x+\delta)^2 - x^2}{\delta} = \lim_{\delta \to 0} \frac{\cancel{x^2} + \delta^2 + 2\delta x - \cancel{x^2}}{\delta} =$$

$$= \lim_{\delta \to 0} \frac{\cancel{\delta}(\delta + 2x)}{\cancel{\delta}} = 2x \qquad \frac{dx^2}{dx} = 2x$$

$$\frac{dx^n}{dx} = n\,x^{n-1} \qquad n = 1, 2, 3, \ldots$$

$$n = 1 \qquad \frac{dx}{dx} = 1$$

$$n = 2 \qquad \frac{dx^2}{dx} = 2x$$

$$f(x) = x^n$$

$$\frac{df}{dx} = \lim_{\delta \to 0} \frac{(x+\delta)^n - x^n}{\delta}$$

sviluppo binomiale

$$(x+\delta)^n = \underbrace{(x+\delta)\cdot(x+\delta)\cdot\ldots\cdot(x+\delta)}_{n \text{ volte}} =$$

$$= x^n + n\delta x^{n-1} + (\dots)\delta^2 + (\dots)\delta^3 \dots$$

$$\frac{df}{dx} = \lim_{\delta \to 0} \frac{n\cancel{\delta}x^{n-1} + (\dots)\delta^{\cancel{2}\,1} + (\dots)\delta^{\cancel{3}\,2} + \dots}{\cancel{\delta}} = n x^{n-1}$$

La formula $\frac{df(x)}{dx} = \underset{0}{n\,x^{n-1}}$ per $f(x) = \underline{\underline{x^n}}$

vale per n qualunque

$$n=0 \qquad f(x) = x^n = 1 \qquad (x \neq 0)$$

$$n=-1 \qquad f(x) = \frac{1}{x} \qquad \frac{df}{dx} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$n=-2 \qquad \frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$ e così via

$$f(x) = \text{sen}\, x$$

$$\text{sen}(\alpha+\beta) = \text{sen}\,\alpha \cos\beta + \text{sen}\,\beta \cos\alpha$$

$$\lim_{\delta \to 0} \frac{\text{sen}\,\delta}{\delta} = 1 \qquad \lim_{\delta \to 0} \frac{1-\cos\delta}{\delta^2} = \frac{1}{2}$$

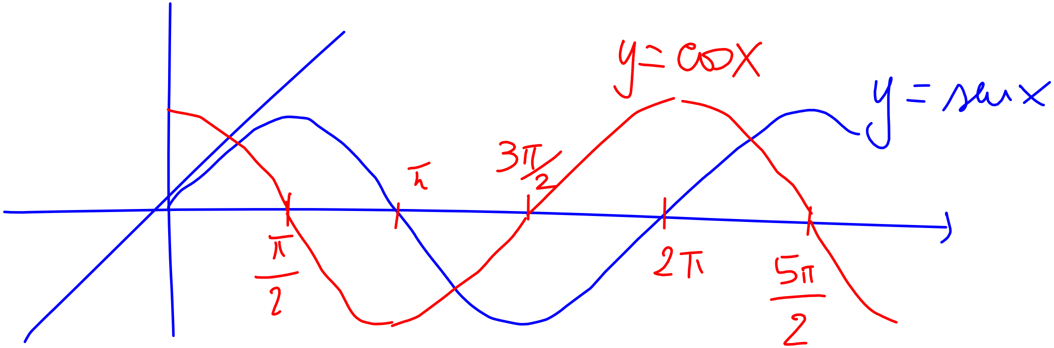
$$\frac{df}{dx} = \lim_{\delta \to 0} \frac{\text{sen}(x+\delta) - \text{sen}(x)}{\delta} =$$

$$= \lim_{\delta \to 0} \frac{\text{sen}\, x \cdot \cos\delta + \cos x \cdot \text{sen}\,\delta - \text{sen}(x)}{\delta} =$$

$$= \lim_{\delta \to 0} \cos x \cdot \underbrace{\left(\frac{\text{sen}\,\delta}{\delta}\right)}_{\to 1} + \lim_{\delta \to 0} \text{sen}\, x \underbrace{\left(\frac{\cos\delta - 1}{\delta^2}\right)}_{\to -\frac{1}{2}} \delta =$$

$$= \cos x$$

$$\frac{d\,\text{sen}\,x}{dx} = \cos x \qquad \frac{d\,\cos x}{dx} = -\,\text{sen}\,x$$



$$\lim_{x \to 0} \frac{\text{sen}\,x}{x} = 1$$

$f''(x)$ = derivata di $f'(x)$

$$\frac{d^2\,\text{sen}\,x}{dx^2} = -\,\text{sen}\,x$$

$$\frac{de^x}{dx} = e^x \qquad f(x) = e^x \qquad e^{a+b} = e^a \cdot e^b$$

$$f'(x) = \lim_{\delta \to 0} \frac{e^{x+\delta} - e^x}{\delta} = \lim_{\delta \to 0} \frac{e^x \cdot e^\delta - e^x}{\delta} =$$

$$= \lim_{\delta \to 0} e^x \underbrace{\frac{e^\delta - 1}{\delta}} = e^x \quad \text{perché} \lim_{\delta \to 0} \frac{e^\delta - 1}{\delta} = 1$$

$$f(x) = \ln x \qquad x > 0 \qquad \frac{d \ln x}{dx} = \frac{1}{x}$$

$$\lim_{\delta \to 0} \frac{\ln(x+\delta) - \ln(x)}{\delta} \qquad \ln(a \cdot b) = \ln a + \ln b$$

$$\ln(x+\delta) = \ln\left[x\cdot\left(1+\frac{\delta}{x}\right)\right] = \ln x + \ln\left(1+\frac{\delta}{x}\right)$$

$x \neq 0$

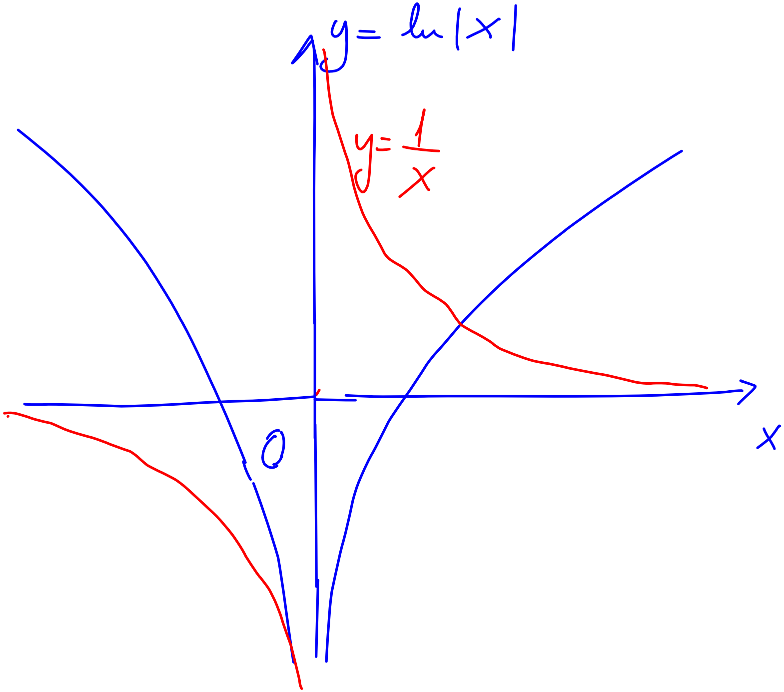
$$\ln(a \cdot b)$$

$$f'(x) = \lim_{\delta\to 0}\frac{\cancel{\ln x} + \ln\left(1+\frac{\delta}{x}\right) - \cancel{\ln x}}{\delta} = \lim_{\delta\to 0}\frac{\ln\left(1+\frac{\delta}{x}\right)}{\delta} =$$

$$\lim_{\delta\to 0}\frac{\ln(1+\delta)}{\delta} = 1$$

$$= \lim_{\frac{\delta}{x}\to 0}\underbrace{\frac{\ln\left(1+\frac{\delta}{x}\right)}{\frac{\delta}{x}}}_{\to 1}\frac{1}{x} = \frac{1}{x}$$

$$\frac{d\ln x}{dx} = \frac{1}{x} \qquad \frac{d\ln|x|}{dx} = \frac{1}{x}$$



$f(x) = \ln|x|$

è una funzione pari

$f(-x) = f(x)$

La derivata di una funzione pari è dispari $f(-x) = -f(x)$

$\frac{1}{x}$ è dispari

Regole:

la derivata della somma è la somma delle derivate

$$\frac{d}{dx}\left(f(x)+g(x)\right)=\frac{df(x)}{dx}+\frac{dg(x)}{dx}$$

se f e g sono entrambe $\underline{\text{derivabili}}$

Se k è costante $\frac{d}{dx}\left(k\,f(x)\right)=k\frac{df}{dx}$

Regola di Leibniz

$$\frac{d}{dx}\left(f(x)\,g(x)\right)=\frac{df(x)}{dx}\,g(x)+f(x)\,\frac{d\,g(x)}{dx}$$

$$\frac{d}{dx}\left(f(x)\,g(x)\right) = \lim_{\delta \to 0} \frac{f(x+\delta)\,g(x+\delta) - f(x)\,g(x)}{\delta} =$$

$$= \lim_{\delta \to 0} \frac{\left(f(x+\delta) - f(x)\right) g(x+\delta) + f(x)\,g(x+\delta) - f(x)\,g(x)}{\delta} =$$

$$= \lim_{\delta \to 0} \underbrace{\boxed{\frac{f(x+\delta) - f(x)}{\delta}}}_{\to \frac{df(x)}{dx}} \underbrace{g(x+\delta)}_{\to g(x)\ *} + \lim_{\delta \to 0} f(x) \underbrace{\frac{g(x+\delta) - g(x)}{\delta}}_{\to \frac{dg(x)}{dx}}$$

$$= \frac{df}{dx}(x)\, g(x) + f(x)\, \frac{dg(x)}{dx}$$

È vero che $\lim_{\delta \to 0} g(x+\delta) = g(x)$?

Sì, se $g(x)$ è continua. Ma siccome $g(x)$ è stata ipotizzata derivabile in x, è anche continua.

Se f è derivabile in x, è continua in x

Infatti, f derivabile vuol dire

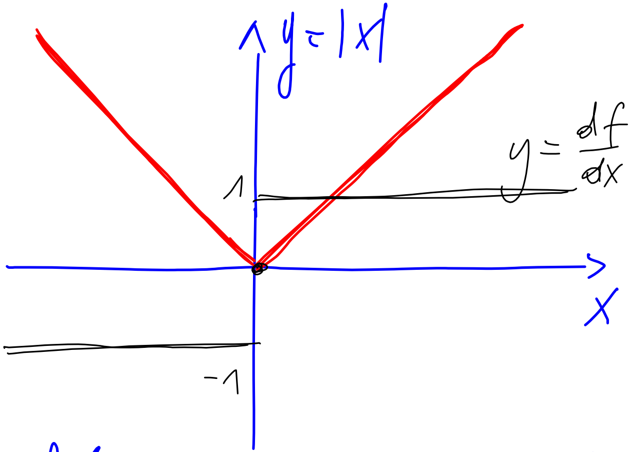
$$\exists \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} = f'(x) = \underline{\underline{\lim_{\delta \to 0} g(\delta)}}$$

Chiamo $g(\delta) = \dfrac{f(x+\delta) - f(x)}{\delta}$

$$\lim_{\delta \to 0} \left(f(x+\delta) - f(x) \right) = \lim_{\delta \to 0} \underset{\uparrow}{\delta} \cdot \underbrace{g(\delta)} = 0$$

cioè $\lim_{\delta \to 0} f(x+\delta) = f(x)$: f è continua in x

Esempio $f(x) = |x| = \begin{cases} x & \text{se } x > 0 \\ 0 & \text{se } x = 0 \\ -x & \text{se } x < 0 \end{cases}$



$f(x)$ è continua

$$\frac{df}{dx} = \begin{cases} 1 & \text{per } x > 0 \\ -1 & \text{per } x < 0 \end{cases}$$

$f(x) = |x|$ non è derivabile in $x = 0$

Derivata della funzione composta

$$y = f(x) \qquad x = g(t) \qquad f(g(t))$$

$$\frac{d}{dt} f(g(t)) = \frac{d\,f(g(t))}{d\,g(t)} \cdot \frac{d\,g(t)}{dt} =$$

= derivata di f come funzione di g · derivata di g

$$\frac{d}{dx} \operatorname{sen}(2x) = \frac{d \operatorname{sen}(2x)}{d\,(2x)} \cdot \frac{d\,2x}{dx} = \cos(2x) \cdot 2$$

$$\frac{d\,\text{sen}(x)}{dx} = \cos x$$

quindi

$$\frac{d\,\text{sen}(g)}{dg} = \cos(g)$$

$$\frac{d\,\text{sen}\,(2x)}{d2x} = \cos(2x)$$

$$\frac{d e^{\boxed{x}}}{d\boxed{x}} = e^{\boxed{x}}$$

$$\frac{d e^{x^2}}{dx} = \underbrace{\frac{d e^{\boxed{x^2}}}{d\boxed{x^2}}}_{} \cdot \frac{dx^2}{dx} =$$

$$= e^{x^2} \cdot 2x = 2x e^{x^2}$$

$$\frac{d}{dx}\text{sen}\left(x^2\right) = \underbrace{\frac{d\,\text{sen}(x^2)}{dx^2}}_{\cos(x^2)} \cdot \frac{dx^2}{dx} = 2x\cos\left(x^2\right)$$

$$\frac{d}{dx}(\operatorname{sen} x)^2 = \frac{d(\operatorname{sen} x)^2}{d \operatorname{sen} x} \cdot \frac{d \operatorname{sen} x}{dx} = 2 \operatorname{sen} x \cos x$$

$$\frac{dx^2}{dx} = 2x \Rightarrow \frac{d(\operatorname{sen} x)^2}{d \operatorname{sen} x} = 2 \operatorname{sen} x$$

$x \to \operatorname{sen} x$

verifica colla regola di Leibniz

$$\frac{d}{dx}(\operatorname{sen} x)^2 = \frac{d}{dx} \operatorname{sen} x \cdot \operatorname{sen} x = \cos x \cdot \operatorname{sen} x +$$

$$+ \cos x \cdot \operatorname{sen} x =$$

$$= 2 \operatorname{sen} x \cos x$$

Derivata del quoziente di funzioni

Se $y = \frac{f(x)}{g(x)}$ $\qquad \frac{dy}{dx} = \frac{f'(x)\,g(x) - g'(x)\,f(x)}{g^2(x)}$

$y = \mathrm{tg}(x) = \frac{\mathrm{sen}\,x}{\cos x}$ $\qquad f = \mathrm{sen}\,x \qquad g = \cos x$

$f' = \cos x \qquad g' = -\mathrm{sen}\,x$

$$\frac{d\,\mathrm{tg}(x)}{dx} = \frac{\cos^2 x + \mathrm{sen}^2 x}{\cos^2(x)} = \frac{1}{\cos^2 x}$$

$(\cos x)^2 \neq \cos(x^2)$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f^2(x)}$$

esempio $f = x$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Derivata della funzione inversa

$y = f(x) \qquad x = f^{-1}(y)$

Scambiano $x \leftrightarrow y$

$$\underline{}$$

$$y = f^{-1}(x)$$

$$f'(x) = \frac{df(x)}{dx} = \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} \longleftarrow$$

derivata della funzione inversa

$$\frac{df^{-1}(y)}{dy} = \frac{1}{\frac{df(x)}{dx}} = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

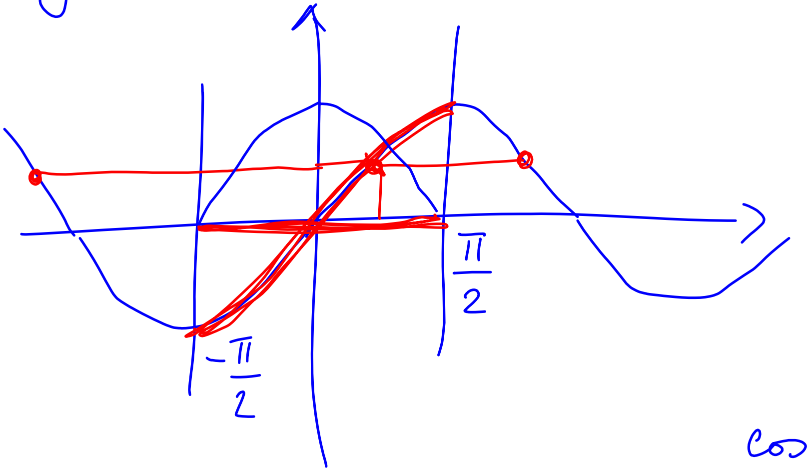
$$y = e^{x} \qquad x = \ln y \; * \qquad y > 0$$

$$\frac{dy}{dx} = e^{x} \qquad \frac{dx}{dy} \overset{*}{=} \frac{1}{\frac{dy}{dx}} = \frac{1}{e^{x}} = \frac{1}{y} \overset{*}{=} \frac{d\ln y}{dy}$$

Scambio $x \leftrightarrow y$ in $*$:

$$y = \ln x \qquad \frac{dy}{dx} = \frac{1}{x} = \frac{d\ln x}{dx} \qquad x > 0$$

$y = \operatorname{sen} x$ $\exists$ inversa per $\boxed{-\frac{\pi}{2} \le x \le \frac{\pi}{2}}$



$x = \operatorname{arcsen} y$

$$\cos^2 x + \operatorname{sen}^2 x = 1$$

$$\cos^2 x = 1 - \operatorname{sen}^2 x = 1 - y^2$$

$$\frac{dy}{dx} = \cos x \qquad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-y^2}}$$

$$\frac{d \operatorname{arcsen} y}{dy} = \frac{1}{\sqrt{1-y^2}} \qquad \frac{d \operatorname{arcsen} x}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$f(x)$	$f'(x)$
k	0
x^n	$n x^{n-1}$
$\text{sen}\, x$	$\cos x$
$\cos x$	$-\text{sen}\, x$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\text{arcsen}\, x$	$\frac{1}{\sqrt{1-x^2}}$
$\text{tg}\, x$	$1/\cos^2 x$

k = costante

$f(x)+g(x)$	$f'(x)+g'(x)$
$k\, f(x)$	$k\, f'(x)$
$f(x)\, g(x)$	$f'(x)\, g(x) + f(x)\, g'(x)$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)\, g(x) - f(x)\, g'(x)}{g^2(x)}$
$f(g(x))$	$\frac{df(g)}{dg} \cdot \frac{dg(x)}{dx}$
$x = f^{-1}(y)$	$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

Limiti di forme indeterminate $\frac{0}{0}$ $\frac{\infty}{\infty}$

Teorema di de l'Hôpital

Se f e g sono continue in I_{x_0} (intorno di x_0) e derivabili in $I_{x_0} \setminus \{x_0\}$ e

$g(x) \neq 0$ $g'(x) \neq 0$ in $I_{x_0} \setminus \{x_0\}$

Se $\frac{f(x)}{g(x)} = \frac{0}{0}$ o $\frac{\infty}{\infty}$ per $x = x_0$

Allora $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$

Esempi: $\quad y = \frac{\operatorname{sen} x}{x} \quad x \to 0$

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = \lim_{x \to 0} \frac{\frac{d \operatorname{sen} x}{dx}}{\frac{dx}{dx}} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\operatorname{sen} x}{2x} = \frac{1}{2}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1$$

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = 1$$

$$\frac{d\,f(g(x))}{dx} = \frac{df}{dg}\;\frac{dg}{dx}$$

$$\frac{d\ln(1+x)}{dx} = \frac{d\ln(1+x)}{d\,(1+x)}\;\frac{d\,(1+x)}{dx} = \frac{1}{1+x}\cdot 1$$

$$\underline{g = 1+x} \qquad \frac{d\ln g}{dg} = \frac{1}{g} \qquad \frac{dg}{dx} = 1$$

$$\frac{d\ln(1+x^2)}{dx} = \frac{d\ln g}{dg}\cdot\frac{dg}{dx} = \frac{1}{1+x^2}\,2x$$

$$g = 1+x^2$$

$$\lim_{x \to 0} \frac{2x+1}{x+1} = 1 \neq \lim_{x \to 0} \frac{\frac{d(2x+1)}{dx}}{\frac{d(x+1)}{dx}} =$$

$$= \lim_{x \to 0} \frac{2}{1} = 2$$

$$\frac{dx^3}{dx} = 3x^2$$

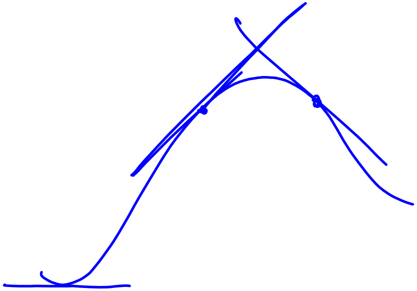
$$\lim_{x \to \infty} \frac{7x^3 + 2x^2 + 3x + 5}{4x^3 + 5x^2 + 3x + 8} = \frac{7}{4} =$$

$$= \lim_{x \to \infty} \frac{21x^2 + 4x + 3}{12x^2 + 10x + 3} = \lim_{x \to \infty} \frac{42x + 4}{24x + 10} =$$

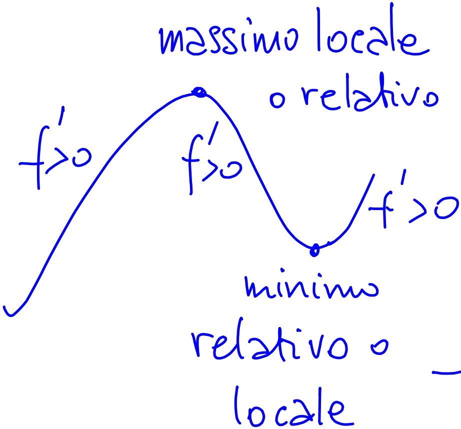
$$= \lim_{x \to \infty} \frac{42}{24} = \frac{\cancel{42}^{7}}{\cancel{24}_{4}} = \frac{7}{4}$$

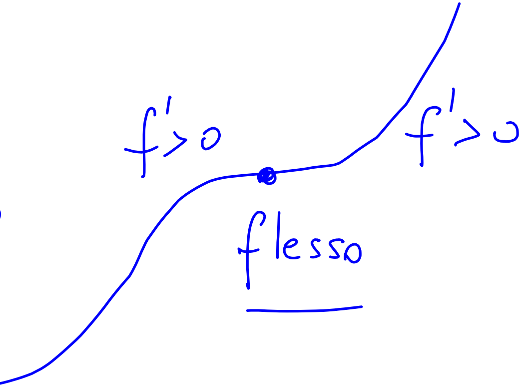
$y = f(x)$

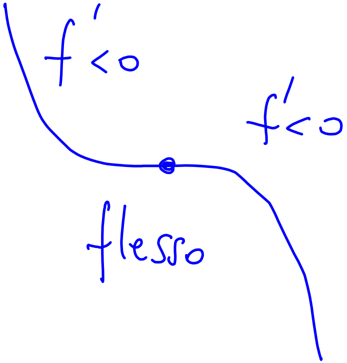
la funzione è crescente dove $f'(x) > 0$ e decrescente dove $f'(x) < 0$



Dove $f'(x) = 0$ si ha un punto stazionario







Punti stazionari : punti x_0 dove $f'(x_0) = 0$

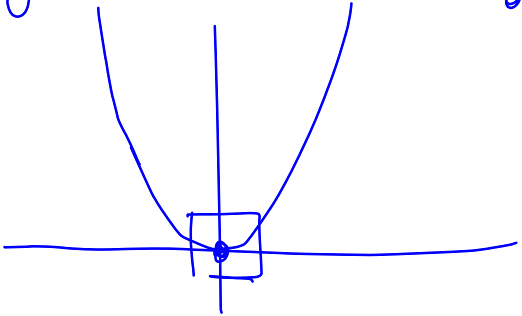
x_0 è un minimo locale se $f''(x_0) > 0$

x_0 è un massimo locale se $f''(x_0) < 0$

Se invece anche $f''(x_0) = 0$ allora bisogna studiare le derivate successive

$y = x^2 = f(x)$

$$\frac{dy}{dx} = 2x = f'(x) = \begin{cases} >0 & \text{per } x>0 \\ <0 & \text{per } x<0 \end{cases}$$



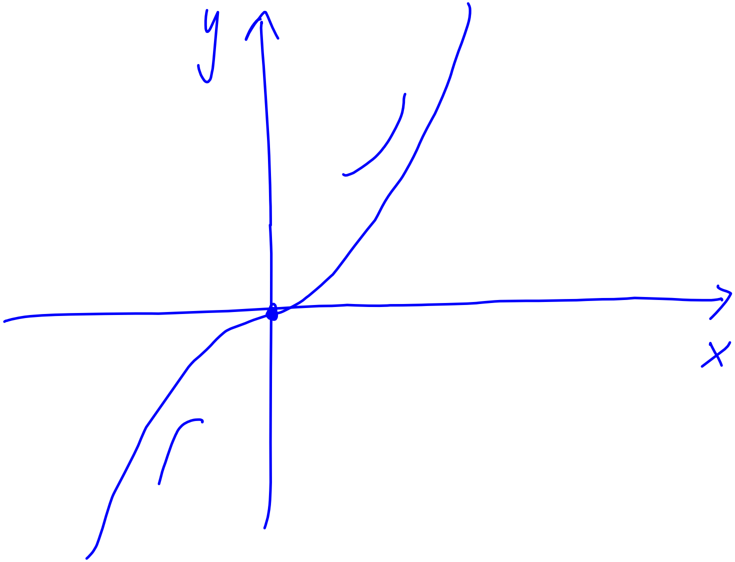
$f'(0) = 0$

$$\frac{d^2y}{dx^2} = f''(x) = 2$$

$y = x^3$ $\quad \frac{dy}{dx} = 3x^2$

sempre >0

tranne che in $x=0$



$x=0$ è un flesso

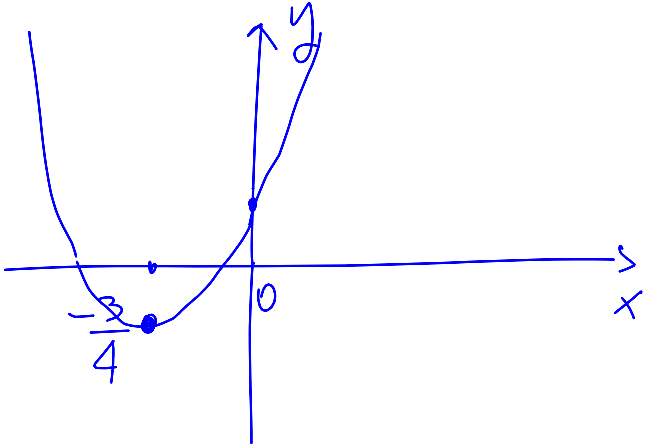
$$\frac{d^2y}{dx^2} = 6x$$

Il segno della derivata seconda indica la concavità/convessità della funzione

$y = 2x^2 + 3x + 1$

$$\frac{d^2y}{dx^2} = 4$$

$$\frac{dy}{dx} = 4x+3 \begin{cases} >0 \text{ per } x > -\frac{3}{4} \\ <0 \text{ per } x < -\frac{3}{4} \\ =0 \text{ per } x = -\frac{3}{4} \end{cases}$$



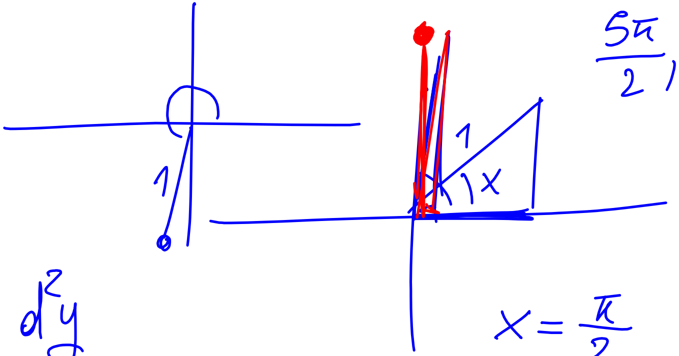
$$y\left(-\frac{3}{4}\right) = \cancel{2}\,\frac{9}{\cancel{16}_{8}} - \frac{9}{4} + 1 =$$

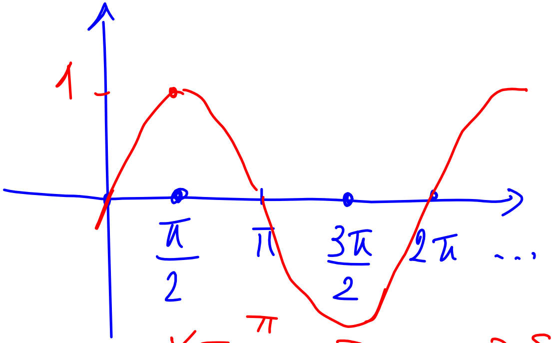
$$= \frac{9-18+8}{8} = -\frac{1}{8}$$

$$y(0) = 1$$

$$y = \operatorname{sen} x \qquad \frac{dy}{dx} = \cos x$$

$$\cos x = 0 \quad \text{per } x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$





$$\frac{d^2y}{dx^2} = -\operatorname{sen} x$$

$$x = \frac{\pi}{2}$$

$$y\left(\frac{\pi}{2}\right) = 1$$

$$y''\left(\frac{\pi}{2}\right) = -1$$

$x = \frac{\pi}{2}$ è un massimo relativo

$$y = \ln(x^2) \qquad \frac{dy}{dx} = \frac{d\ln(x^2)}{dx} = \frac{d\ln(x^2)}{dx^2} \cdot \frac{dx^2}{dx} =$$

$$\frac{d\ln(x)}{dx} = \frac{1}{x} \Rightarrow \frac{d\ln(x^2)}{dx^2} = \frac{1}{x^2} \qquad = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$$

$$\to \ln|x|^a = a\ln|x| \qquad a > 0$$

$$\ln x^2 = 2\ln|x| \qquad \frac{de^x}{dx} = e^x$$

$$\frac{da^x}{dx} = ? \qquad a^x = e^{\ln(a^x)}$$

$$\ln a^x = x\ln a$$

↙

$$a^x = e^{x\ln a}$$

$$a = e^{\ln a}$$

$$\underline{\ln e^a = a}$$

$$\frac{d\,a^x}{dx} = \frac{d\,e^{x\ln a}}{dx} = \frac{d\,e^{x\ln a}}{d(x\ln a)} \cdot \frac{d(x\ln a)}{dx} =$$

$$\downarrow$$

$$= e^{x\ln a} \cdot \ln a =$$

$$= a^x \cdot \ln a$$

$$\boxed{\frac{de^x}{dx} = e^x}$$

$$\downarrow$$

$$\frac{d\,e^{x^2}}{dx} = \frac{d\,e^{x^2}}{dx^2} \cdot \frac{dx^2}{dx} = e^{x^2} \cdot 2x$$

$$\frac{d}{dx}\cos\left(e^{2x}\right) = \frac{d\cos(e^{2x})}{de^{2x}} \cdot \frac{de^{2x}}{d(2x)} \cdot \frac{d(2x)}{dx} =$$

$$= -\operatorname{sen}\left(e^{2x}\right) \cdot e^{2x} \cdot 2$$

$$\frac{d\cos x}{dx} = -\operatorname{sen} x \qquad \frac{de^x}{dx} = e^x \qquad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{1-\cos(x^2)}{x^3 \operatorname{sen} x} = \lim_{x\to 0} \frac{1-\cos(x^2)}{x^4\left(\frac{\operatorname{sen} x}{x}\right)} =$$

$$= \lim_{x\to 0} \frac{1-\cos(x^2)}{x^4} = \frac{1}{2}$$

Con l'Hôpital:

$$\lim_{x \to 0} \frac{1 - \cos(x^2)}{x^3 \operatorname{sen} x} = \lim_{x \to 0} \frac{\operatorname{sen}(x^2) \cdot 2\cancel{x}}{3x^{\cancel{2}} \operatorname{sen}(x) + x^{\cancel{3}2} \cos(x)} =$$

$$= \lim_{x \to 0} \frac{2 \operatorname{sen}(x^2)}{\underline{3x}\, \underline{\operatorname{sen}}(x) + \underline{x^2}\, \underline{\cos}(x)} =$$

$$= \lim_{x \to 0} \frac{2 \cos(x^2) \cdot 2x}{3 \operatorname{sen}(x) + 3x \cos(x) + 2x \cos(x) - x^2 \operatorname{sen}(x)} =$$

$$= \lim_{x \to 0} \frac{4x \cos(x^2)}{3 \operatorname{sen}(x) + 5x \cos(x) - x^2 \operatorname{sen}(x)}$$

$$= \lim_{x \to 0} \frac{4\cos(x^2)}{\underbrace{3\,\frac{\text{sen}(x)}{x}}_{\to 3} + \underbrace{5\cos(x)}_{\to 5} - \underbrace{x\,\text{sen}(x)}_{\to 0}} = \frac{4}{8} = \frac{1}{2}$$

divido per x
sopra e sotto

$$\lim_{x \to 0} \frac{\text{sen}\,x}{x} = 1$$

$$\lim_{x \to \infty} \frac{3x^5 + 6}{4x^5 + x^4} = \frac{3}{4}$$

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

$$\lim_{x \to 0} \frac{\text{sen}(2x)}{\ln(1+2x)} = \lim_{x \to 0} \frac{\text{sen}(2x)}{2x} \, \frac{2x}{\ln(1+2x)} = 1$$

Derivate: $f(x) = \underbrace{2x}\,\underline{\text{sen}(3x^2)}$

$$f'(x) = 2\,\text{sen}(3x^2) + 2x \cdot 6x \cos(3x^2) =$$

$$\frac{d\,\text{sen}(3x^2)}{dx} = \cos(3x^2)\,6x$$

$$= 2\,\text{sen}(3x^2) +$$

$$+ 12x^2 \cos(3x^2)$$

$$y = f(x) = \underbrace{x^2}\,e^{x^3}$$

$$f'(x) = 2x\,e^{x^3} + x^2 \cdot e^{x^3} \cdot 3x^2 =$$

$$= 2x\,e^{x^3} + 3x^4 e^{x^3}$$

Studiare $y = \dfrac{x^2}{\underline{\underline{x-1}}}$ Dominio: $\mathbb{R} \setminus \{1\}$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to \infty} \frac{2x}{1} = +\infty$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

$$\lim_{x \to 1^+} f(x) = +\infty$$

limite destro

$$\lim_{x \to 1^-} f(x) = -\infty$$

limite sinistro

$$f'(x) = \frac{2x(x-1) - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} =$$

$$= \frac{x(x-2)}{(x-1)^2}$$

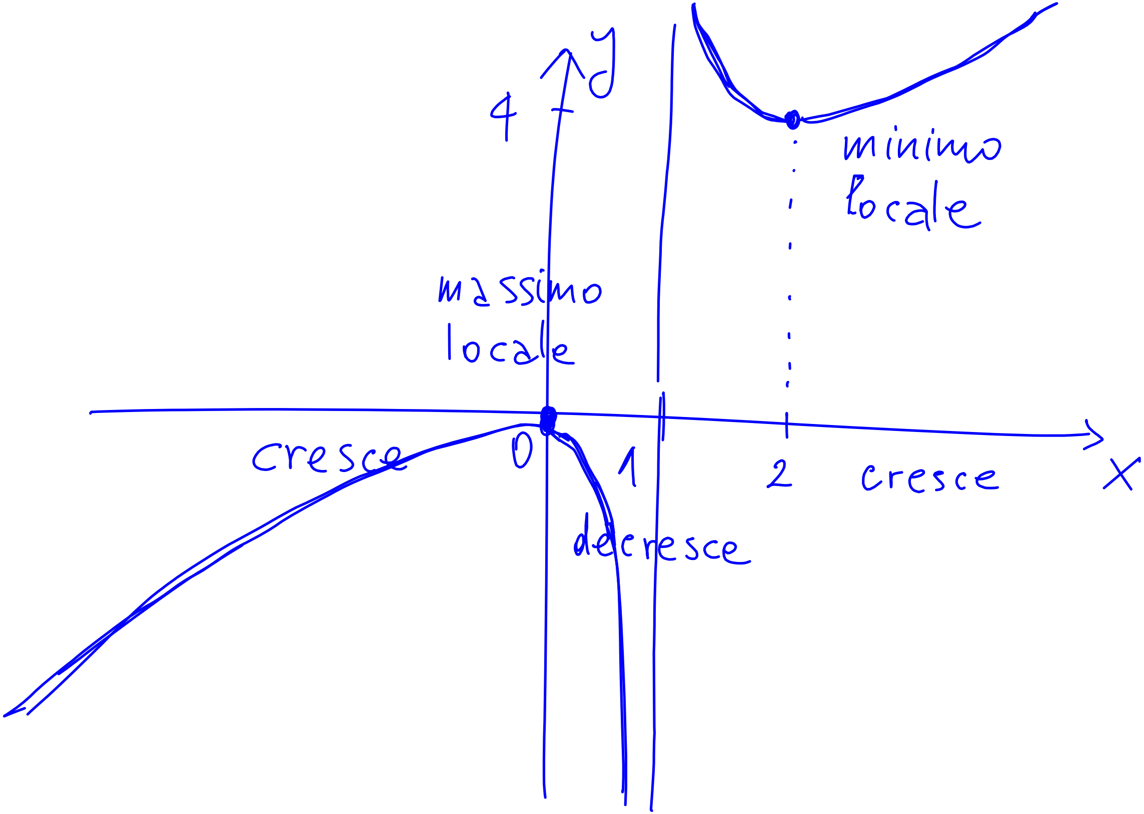
$f(0) = 0$

$f(2) = 4$

$f'(x) = 0$ per $x = 0$ $x = 2$

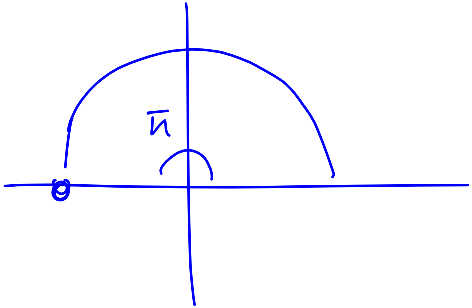
$f'(x) > 0$ per $\begin{cases} x > 0 & x > 2 \\ x < 0 & x < 2 \end{cases} = \begin{cases} x > 2 \\ x < 0 \end{cases}$

$f'(x) < 0$ per $0 < x < 2$


4
y
massimo
locale
minimo
locale
cresce
0
1
2
cresce
X
decresce

$$\lim_{x\to 1} \frac{\operatorname{sen}(\pi x)}{\ln(x)} = \lim_{x\to 1} \frac{\cos(\pi x)\cdot \pi}{\frac{1}{x}} = -\pi$$

$$\frac{0}{0}$$



$$\lim_{x\to 0} \frac{x \operatorname{sen} x}{(e^{2x}-1)^2} = \lim_{x\to 0} \frac{\operatorname{sen} x + x\cos x}{2(e^{2x}-1)\cdot e^{2x}\cdot 2} =$$

$$= \frac{1}{4} \lim_{x\to 0} \frac{\operatorname{sen}(x) + x\cos(x)}{e^{2x}-1} =$$

$$= \frac{1}{4}\lim_{x\to 0}\frac{\cos(x)+\cos(x)-x\,\mathrm{sen}(x)}{2e^{2x}} =$$

$$= \frac{1}{4}\,\frac{\cancel{2}}{\cancel{2}} = \frac{1}{4}$$

$$\lim_{x\to 0}\frac{e^{x}-1}{x} = 1$$

$$\lim_{x\to 0}\frac{x\,\mathrm{sen}\,x}{\left(e^{2x}-1\right)^{2}} = \lim_{x\to 0}\frac{2x}{e^{2x}-1}\cdot\frac{2x}{e^{2x}-1}\cdot$$

$$\cdot\frac{\cancel{x}}{2\cancel{x}}\cdot\frac{\mathrm{sen}\,x}{2x} = \frac{1}{4}$$

$$f(x) = \ln|\cos(x)| \qquad \boxed{\frac{d\ln|x|}{dx} = \frac{1}{x}} \leftarrow$$

$$f'(x) = \frac{d\ln|\cos(x)|}{dx} = \frac{d\ln|\cos(x)|}{d\cos x} \cdot \frac{d\cos x}{dx} =$$

$$= \frac{1}{\cos x} \cdot (-\operatorname{sen} x) = -\operatorname{tg}(x)$$

$$f(x) = x \cdot \operatorname{sen}(e^x)$$

$$f'(x) = \operatorname{sen}(e^x) + x \cdot \cos(e^x) \cdot e^x$$

Studiare $y = f(x) = x \underbrace{e^{-\frac{x^2}{2}}}$ Dominio: $\mathbb{R}$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{\cancel{x}}{e^{\frac{\cancel{x}^2}{2}}} = \lim_{x \to \pm\infty} \frac{1}{e^{\frac{x^2}{2}} \cdot x} = 0$$

$$3^{-2} = \frac{1}{3^{+2}} \qquad \frac{d\left(\frac{x^2}{2}\right)}{dx} = x$$

$$f'(x) = e^{-\frac{x^2}{2}} + x \cdot e^{-\frac{x^2}{2}} \cdot (-x) =$$

$$= e^{-\frac{x^2}{2}} \underbrace{(1 - x^2)}$$

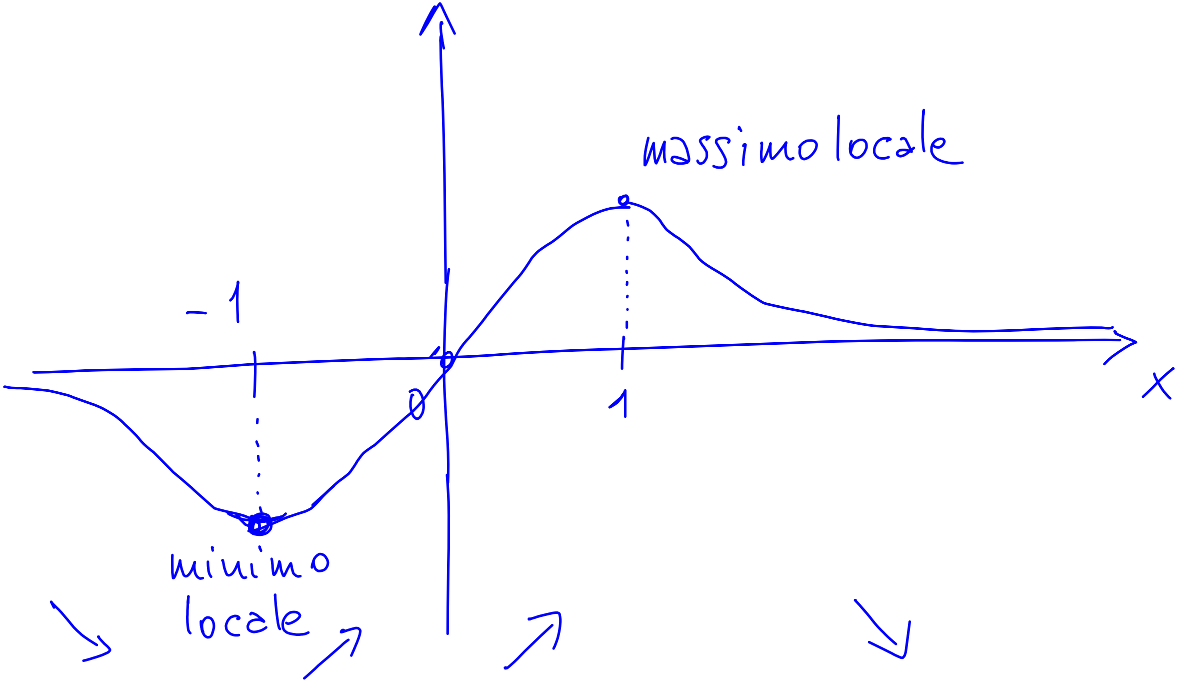
$f'(x) > 0$ per $\underline{-1 < x < 1}$

$f'(x) = 0$ per $x = \pm 1$

$f'(x) < 0$ $x > 1$
$x < -1$

$$f'(x) = e^{-\frac{x^2}{2}}(1-x)(1+x)$$

$$f(1) = e^{-1/2} = \frac{1}{\sqrt{e}} \qquad f(-1) = -\frac{1}{\sqrt{e}}$$



Sviluppo di Taylor

Sia $f(x)$ una funzione infinitamente derivabile

Esempio: $f(x) = e^x$ $\qquad f^{(n)}(x) = e^x$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f'(x)}{1} = \underline{\underline{f'(0)}}$$

$$\downarrow$$

l'Hopital

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0)$$

$$\lim_{x\to 0} \frac{f(x) - f(0) - x f'(0)]}{x} = \lim_{x\to 0} \frac{f'(x) - f'(0)}{1} = 0$$

Se $f(x) \to 0$ $g(x) \to 0$ e $\frac{f(x)]}{g(x)]} \to 0$

Si dice che f è un infinitesimo di ordine superiore a g e si scrive $f = o(g)$

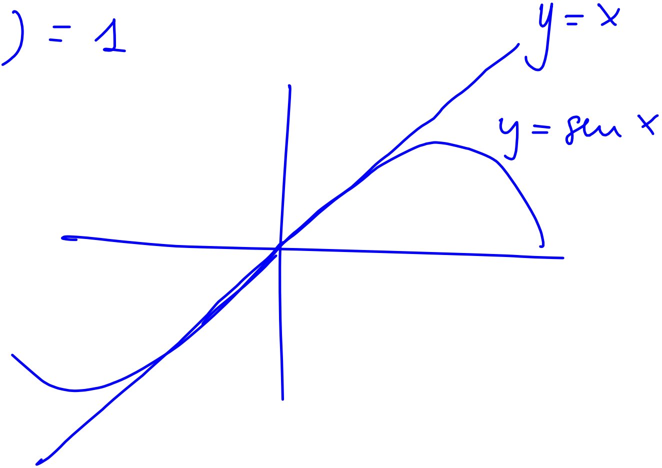
$$f(x) - f(0) - x f'(0) = o(x)$$ "o piccolo"

Vuol dire $\lim_{x\to 0} \frac{o(x)}{x} = 0$

$$f(x) = \underline{f(0)} + \underline{x f'(0)} + \underbrace{o(x)}$$

$$f(x) = \operatorname{sen} x \qquad f(0) = 0 \qquad f'(x) = \cos x$$

$$f'(0) = 1$$



$$\operatorname{sen}(x) = x + o(x)$$

dividiamo per x

$$\frac{\operatorname{sen}(x)}{x} = \underbrace{1} + \underbrace{\frac{o(x)}{x}}_{\text{tende a 0 per } x \to 0}$$

vuol dire $\frac{\operatorname{sen} x}{x} \to 1$ per $x \to 0$

Serie di Taylor attorno a $x=0$

$$f(x) = f(0) + \frac{x}{1!} f^{(1)}(0) + \frac{x^2}{2!} f^{(2)}(0) + \underbrace{+ \frac{x^3}{3!} f^{(3)}(0) + \dots}$$

$$f(x) = \underbrace{f(0)} + \underbrace{x f'(0)} + \underbrace{\frac{x^2}{2} f''(0)} + \underbrace{o(x^2)}$$

Cioè
$$\lim_{x \to 0} \frac{o(x^2)}{x^2} = 0 = \lim_{x \to 0} \frac{f(x) - f(0) - x f'(0) - \frac{x^2}{2} f''(0)}{x^2} =$$

$$\underset{\text{l'Hopital}}{=} \lim_{x \to 0} \frac{f'(x) - f'(0) - x f''(0)}{2x} =$$

$$\underset{\text{l'Hopital}}{=} \lim_{x \to 0} \frac{f''(x) - f''(0)}{2} = 0$$

$f(x) = e^x \qquad f(0) = 1$

$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$

Sviluppo di Taylor dell'esponenziale

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{\underset{6}{3!}} + \frac{x^4}{\underset{24}{4!}} \cdots\cdot$$

Sviluppo di Taylor attorno a x_0 :

$$f(x) = f(x_0) + (x-x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) +$$
$$+ \frac{(x-x_0)^3}{3!} f^{(3)}(x_0) + \cdots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + o((x-x_0)^n)$$

Integrali

Se $f(x)$ ha derivata $f'(x)$ si dice che $f(x)$ è <u>una</u> primitiva di $f'(x)$

(o integrale indefinito)

La primitiva di $f(x)$ è una funzione $F(x)$ tale che

$$\frac{dF(x)}{dx} = f(x)$$

$f(x)$	$F(x)$
0	C
→ x^n	$\frac{x^{n+1}}{n+1} + C$
$\cos x$	$\operatorname{sen} x + C$
$\operatorname{sen} x$	$-\cos x + C$
e^x	$e^x + C$
→ $\frac{1}{x}$	$\ln x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arcsen}(x) + C$
$1/\cos^2 x$	$\operatorname{tg} x + C$

C = costante indeterminata

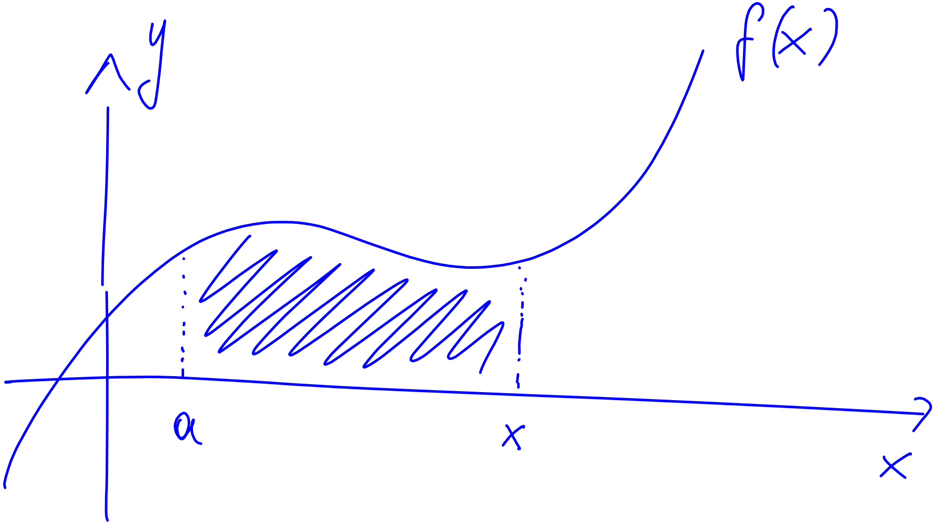
$$n x^{n-1} \quad \cdots \quad x^n + C$$

$$\boxed{x^{n-1} \; \leftarrow\cdots\rightarrow \; \frac{x^n}{n} + C}$$

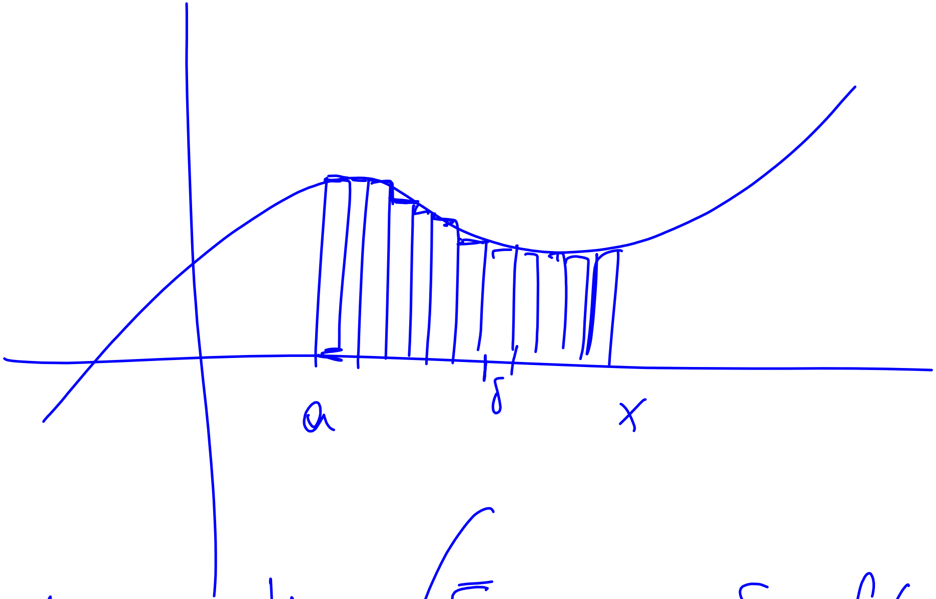
$$n \to n+1$$

$$x^n \; \leftarrow\cdots\rightarrow \; \frac{x^{n+1}}{n+1} + c$$

Significato dell'integrale .... SOMMA AREA



Una primitiva di $f(x)$ è l'area sottesa dal grafico della funzione $f(x)$ compreso tra a (qualunque) e x



Σ somma

$\int$ integrale

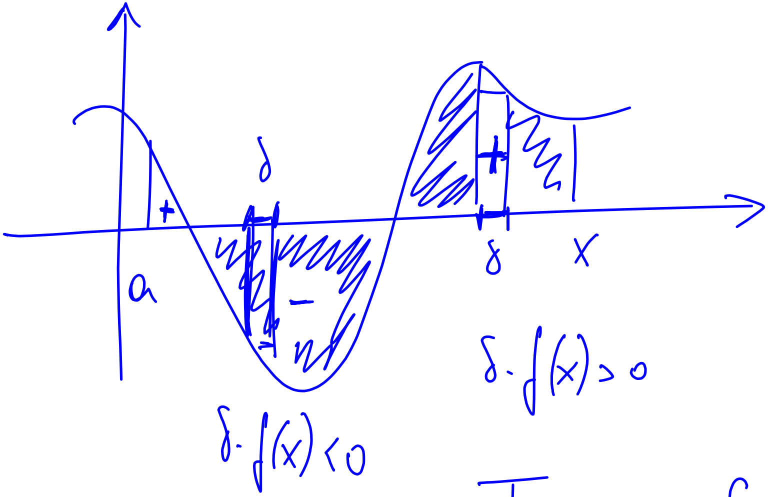
$$\text{Area} = \lim_{\delta \to 0} \left(\text{Somma} \quad \underbrace{\delta}_{\text{base}} \cdot \underbrace{f(x)}_{\text{altezza}} \right) =$$

x ← estremo di integrazione

$$= \int_a^x f(t)\, dt$$

← variabile di integrazione

a ← estremo di integrazione



l'area sottesa dal grafico va intesa in senso algebrico (±)

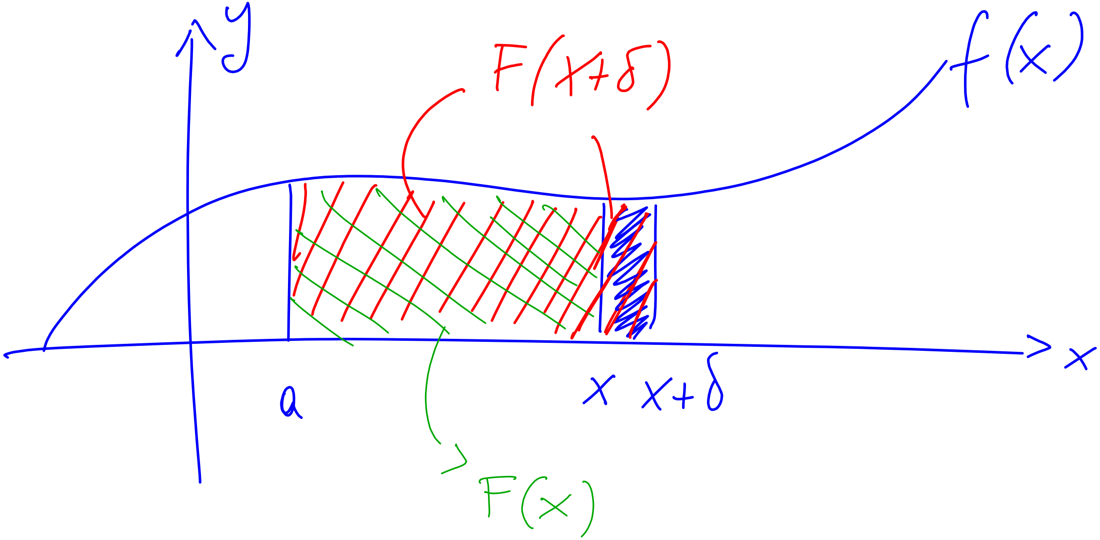
Teorema fondamentale del calcolo integrale

Sia $F(x)$ l'area del grafico sotteso dalla funzione continua $f(x)$ tra a e x. Allora $\frac{dF(x)}{dx} = f(x)$.

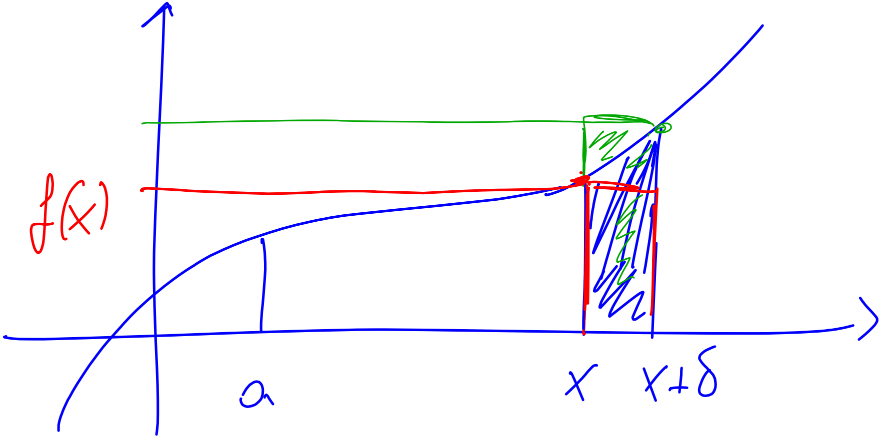
Dimostrazione: $F'(x) = \lim_{\delta \to 0} \frac{F(x+\delta) - F(x)}{\delta}$

$F(x+\delta)$: area tra a e $x+\delta$

$F(x)$: area tra a e x



$F(x+\delta) - F(x)$ = area compresa tra x e $x+\delta$



Studiare il limite $\dfrac{F(x+\delta)-F(x)}{\delta}$ per $\delta \to 0$

Area rettangolo rosso $\leq$ Area $F(x+\delta)-F(x)$ $\leq$ Area rettangolo verde

$\| \qquad\qquad\qquad\qquad\qquad\qquad \|$

$\delta \cdot f(x) \qquad\qquad\qquad\qquad \delta \cdot f(x+\delta)$

$$\delta \cdot f(x) \le F(x+\delta) - F(x) \le \delta \cdot f(x+\delta)$$

Divido per δ

$$\underset{=}{f(x)} \le \underbrace{\frac{F(x+\delta)-F(x)}{\delta}}_{} \le f(x+\delta)$$

$$\lim_{\delta \to 0} f(x+\delta) = f(x)$$

[f continua!!]

Il teorema del confronto $\Rightarrow$

anche $$\lim_{\delta \to 0} \frac{F(x+\delta)-F(x)}{\delta} = f(x) = F'(x)$$

$$\int_a^b f(x)dx =$$ Area del grafico sotteso da $f(x)$ tra a e b

Integrale definito

Proprietà

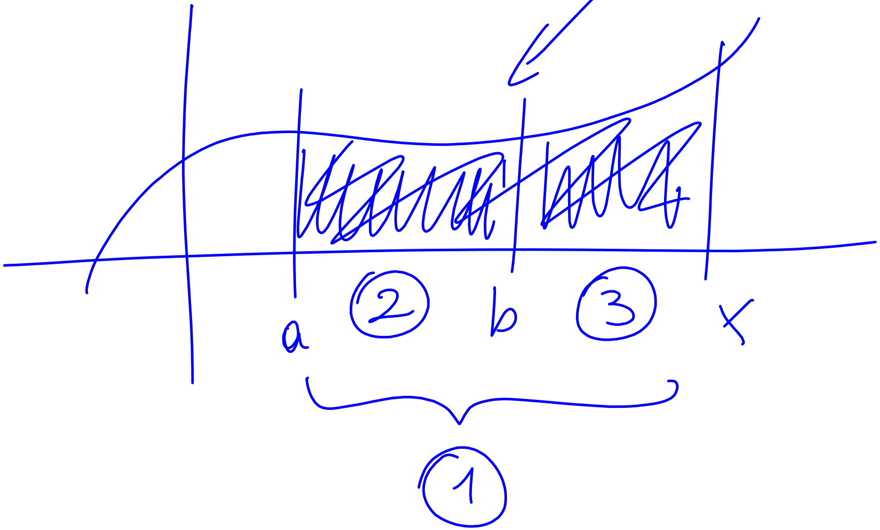
$$\int_a^x \big(f(t) + g(t)\big)dt = \int_a^x f(t)dt + \int_a^x g(t)dt$$

Se k è costante

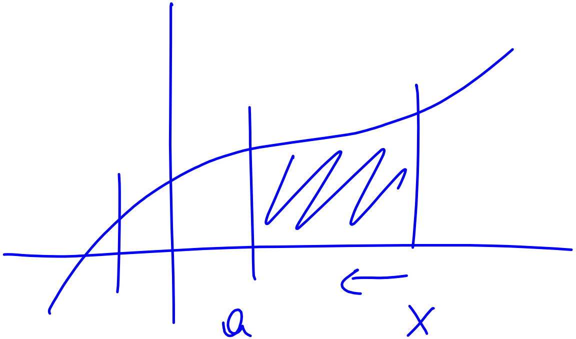
$$\int_a^x \big(k\, f(t)dt\big) = k \int_a^x f(t)dt$$

$$\int_a^a f(t)\,dt = 0$$

$$\underbrace{\int_a^x f(t)\,dt}_{①} = \underbrace{\int_a^b f(t)\,dt}_{②} + \underbrace{\int_b^x f(t)\,dt}_{③} \quad \forall b$$



$$\int_a^x f(t)\,dt = -\int_x^a f(t)\,dt$$



$$\underbrace{\int_a^x f(t)\,dt}_{\text{primitiva } 1} = \underbrace{\int_a^b f(t)\,dt}_{c} + \underbrace{\int_b^x f(t)\,dt}_{\text{primitiva}} \quad \forall b$$

La differenza tra due primitive della stessa funzione è una costante

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

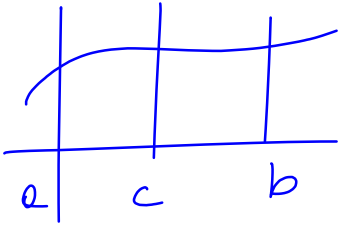
dove F è qualunque primitiva di f

$$\int_c^x f(t)\,dt = F(x) \quad \text{primitiva}$$

$$F(b) = \int_c^b f(t)\,dt \qquad F(a) = \int_c^a f(t)\,dt$$

$$\underline{\underline{F(b)}} - \underline{\underline{F(a)}} = \int_c^b f(t)\,dt - \underline{\int_c^a f(t)\,dt} =$$

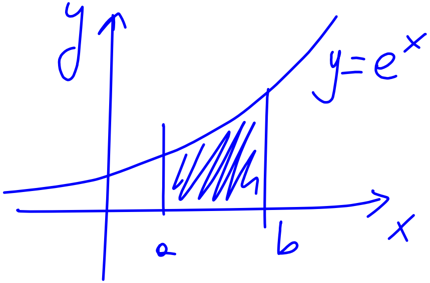
$$= \int_a^c f(t)\,dt + \int_c^b f(t)\,dt = \int_a^b f(t)\,dt$$



$$\int_a^b e^x\,dx = F(b) - F(a)$$

dove $F(x) = e^x$

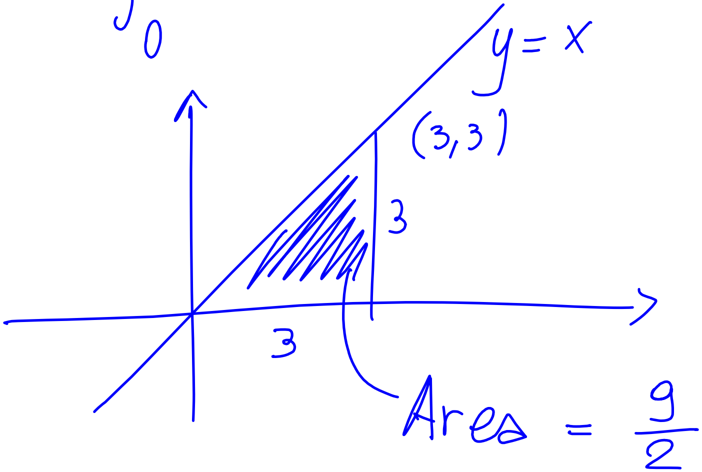
$$= e^b - e^a$$



$$\int_a^b \frac{dx}{x} = F(b)-F(a) = \ln b - \ln a = \ln \frac{b}{a}$$

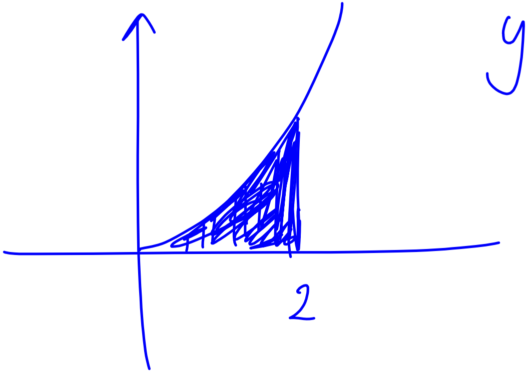
$f(x) = \frac{1}{x}$ $\qquad$ $F(x) = \ln x$

$$\int_0^3 x\,dx = F(3)-F(0) = \frac{3^2}{2} - 0 = \frac{9}{2}$$



$f(x) = x$

$F(x) = \frac{x^2}{2}$



$y = x^2$

$f(x) = x^2 \qquad F(x) = \frac{x^3}{3}$

$$\int_0^2 x^2 \, dx = F(2) - F(0)$$

$$= \frac{2^3}{3} - 0 = \frac{8}{3}$$

Derivata della funzione composta

$$\frac{d f(g(x))}{dx} = \frac{d f(g(x))}{d \, g(x)} \cdot \frac{d g(x)}{dx}$$

Primitiva di $\boxed{2x} \cdot \underbrace{e^{\boxed{x^2}}} \longrightarrow \underline{\underline{e^{x^2}}}$

$2x \cdot \operatorname{sen}(x^2)$ Primitiva $-\cos(x^2)$

$f(x) = e^x \cdot \underline{\operatorname{sen}(e^x)}$ $F(x) = -\underbrace{\cos(e^x)}$

$f(x) = \frac{\cos(x)}{\operatorname{sen}(x)} = \cos(x)\frac{1}{\operatorname{sen}(x)}$ $F(x) = \ln(\operatorname{sen}(x))$

$F'(x) = \frac{1}{\operatorname{sen}(x)} \cos(x)$

$f(x) = 1 \cdot \operatorname{sen}(x+1)$ $F(x) = -\cos(x+1)$

Integrale per sostituzione

$$f(x) = \underbrace{\boxed{2x}}\cdot\underbrace{\text{sen}(x^2)}$$

$\int f(x)\,dx$ si intende

$=$ la primitiva

$$\frac{dg(x)}{dx} f(g(x))$$

$$u = x^2 \qquad \frac{du}{dx} = 2x$$

$$\int f(x)\,dx = \int 2x\,\text{sen}(x^2)\,dx =$$

$$= \int \frac{du}{\cancel{dx}}\,\text{sen}(u)\,\cancel{dx} = \int \text{sen}(u)\,du$$

$$\longrightarrow -\cos(u) \longrightarrow -\cos(x^2)$$

Integrale per parti

$$\frac{d}{dx}\left(f(x)\,g(x)\right) = \frac{df(x)}{dx}\,g(x) + f(x)\,\frac{dg}{dx}$$

primitive $\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$

$$f(x)\,g(x) = P_1 + P_2$$

La primitiva di $\frac{df(x)}{dx}$ è $f(x) + C$

$$\int \frac{df(x)}{\cancel{dx}}\,\cancel{dx} = \int 1 \cdot d\underline{\underline{f}} = f(x)$$

$$\int \frac{df(x)}{dx} g(x)\, dx = \int dx \left[\frac{d}{dx} \left(f(x) g(x) \right) + \right.$$

$$\left. - f(x) \frac{dg(x)}{dx} \right] = \int dx \frac{d}{dx} \left(f(x) g(x) \right) +$$

$$- \int dx\, f(x) \frac{dg(x)}{dx} =$$

$$= f(x) g(x) - \int dx\, f(x) \frac{dg(x)}{dx}$$

Esempi

$$\int x \, \mathrm{sen}(x) \, dx = \int \overset{g(x)}{x} \; \overset{f(x)}{\frac{d(-\cos x)}{dx}} \, dx =$$

$$= -x \cos(x) - \int dx \; 1 \cdot (-\cos x) =$$

$$= -x \cos(x) + \mathrm{sen}(x)$$

Verifica.

$$\frac{d}{dx}\Big(\mathrm{sen}(x) - x\cos(x)\Big) = \cancel{\cos(x)} - \cancel{\cos(x)} + x\,\mathrm{sen}(x)$$

$$\frac{1}{2}\int \boxed{2x}\, sen\underset{u}{(x^2)}\, dx = \frac{1}{2}\int \frac{du}{\cancel{dx}}\, sen(u)\, \cancel{dx} =$$

$$= -\frac{1}{2}\cos(u) = -\frac{1}{2}\cos(x^2)$$

$$\int \ln(x)\, dx = \int 1 \cdot \ln(x)\, dx = \int \frac{d\overset{f}{\boxed{x}}}{dx}\, \underline{\underline{\overbrace{\ln(x)}^{g}}}\, dx =$$

$$= x\ln(x) - \int dx\, \cancel{x} \cdot \frac{1}{\cancel{x}} =$$

$$= x\ln(x) - \int dx\; 1 = x\ln(x) - x$$

$$\int e^{x}\,dx = e^{x}$$

$$2x = u \qquad \frac{du}{dx}$$

$$\int e^{2x}\,dx = \frac{1}{2}\int \underbrace{2}\, e^{2x}\,dx =$$

$$= \frac{1}{2}\int \frac{du}{\cancel{dx}}\, e^{u}\,\cancel{dx} = \frac{1}{2}\int du\, e^{u} =$$

$$= \frac{e^{u}}{2} = \frac{e^{2x}}{2}$$

$$\int e^{3x}\,dx = \frac{e^{3x}}{3}$$

$$\int x e^{x} dx = \int \underbrace{\overset{g}{\boxed{x}} \frac{d \overset{f}{\boxed{e^{x}}}}{dx} dx}_{} =$$

$$= x e^{x} - \int dx \; e^{x} \cdot 1 = x e^{x} - e^{x}$$

$$\frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{\frac{1}{2} - 1} = \frac{1}{2} x^{-\frac{1}{2}} =$$

$$= \frac{1}{2\sqrt{x}}$$

$$\frac{d x^{n}}{dx} = n x^{n-1}$$

$$\frac{d}{dx}\sqrt{x+2} = \frac{1}{2}\,\frac{1}{\sqrt{x+2}}$$

$$\boxed{x+2=u}$$

$$\frac{du}{dx} = 1$$

$$\frac{d\sqrt{u}}{dx} = \boxed{\frac{d\sqrt{u}}{du}}\,\frac{du}{dx} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{x+2}}$$

$$\frac{d}{dx}\sqrt{1+2x^3} = \frac{d\sqrt{u}}{dx} = \frac{d\sqrt{u}}{du}\,\frac{du}{dx} =$$

$$u = 1+2x^3$$

$$\frac{du}{dx} = 6x^2$$

$$= \frac{1}{\cancel{2}\sqrt{u}}\,\overset{3}{\cancel{6}}x^2 = \frac{3x^2}{\sqrt{1+2x^3}}$$

$$\int 7x^2 e^{8x^3} dx = \int e^u \cdot 7 \cdot \frac{1}{24} \frac{du}{\cancel{dx}} \cancel{dx} =$$

$$u = 8x^3$$

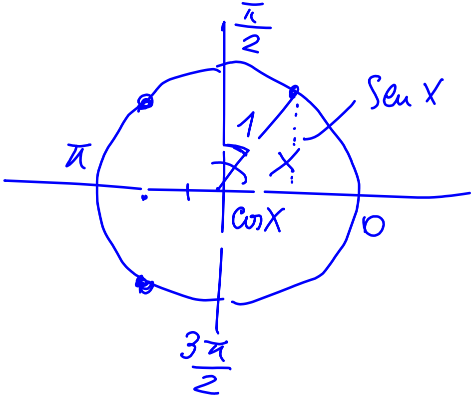
$$\frac{du}{dx} = 24x^2$$

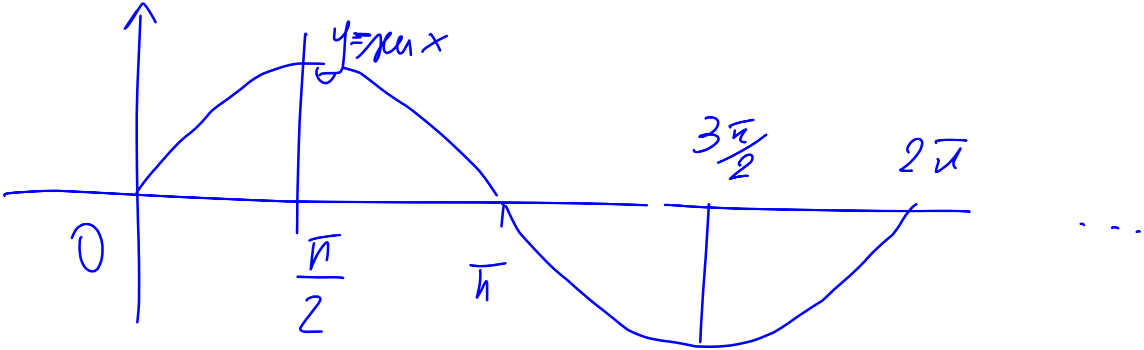
$$= \frac{7}{24}\int e^u du = \frac{7}{24} e^u = \frac{7}{24} e^{8x^3}$$

$f(x) = \text{sen}(x) \qquad x \in [0, 2\pi]$

$f(x) > 0$ dove? $x \in (0, \pi)$

$f(x) < 0$ dove? $x \in (\pi, 2\pi)$





$f(x) = \text{sen}(3x) \qquad x \in \left[0, \frac{2\pi}{3}\right]$

$f(x) > 0$ per ? $\qquad 3x \in (0, \pi) \qquad x \in \left(0, \frac{\pi}{3}\right)$

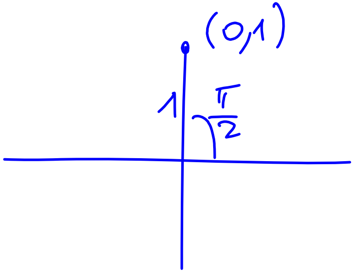
$\frac{0}{3} < \frac{\cancel{3}x}{3} < \frac{\pi}{3} \qquad 0 < x < \frac{\pi}{3}$

Dove $\text{sen}(x^3) > 0$? $\qquad x^3 \in (0, \pi) \qquad 0 < x^3 < \pi$

$0 < x^3$ $\quad x^3 < \pi$ $\quad \text{sen}(x^3) > 0$

$(0)^{1/3} < (x^3)^{1/3}$ $\quad x < \pi^{\frac{1}{3}}$ $\quad$ per $0 < x < \sqrt[3]{\pi}$

$0 < x$



$$\frac{d}{dx}\frac{\text{sen}\,x}{x} = \frac{\cos x \cdot x - \text{sen}\,x}{x^2}$$

$$\lim_{x \to 1} \frac{\cos\left(\frac{\pi}{2}x\right)}{x-1} = \lim_{x \to 1} \frac{-\text{sen}\left(\frac{\pi}{2}x\right) \cdot \frac{\pi}{2}}{1} = -\frac{\pi}{2}$$

l'Hopital

$$\lim_{x \to +\infty} x^3 e^{-\frac{x^2}{2}} = \lim_{x \to +\infty} \frac{x^3}{e^{\frac{x^2}{2}}} \underset{\text{l'Hopital}}{=}$$

$$= \lim_{x \to \infty} \frac{3x^{\cancel{2}}}{\cancel{x}\, \underbrace{e^{\frac{x^2}{2}}}_{}} \underset{\text{l'Hopital}}{=} \lim_{x \to \infty} \frac{3}{x e^{\frac{x^2}{2}}} =$$

$$= 0$$

$$\frac{d}{dx}\left(x^3 e^{-x^2}\right) = 3x^2 e^{-x^2} + x^3 e^{-x^2} \cdot (-2x) =$$

$$= x^2 e^{-x^2}\left(3 - 2x^2\right)$$

$$k'(x) = \cot(x) = \frac{\cos(x)}{\operatorname{sen}(x)}$$

$$\int k'(x)\,dx = \int \frac{\cos(x)}{\operatorname{sen}(x)}\,dx = \int \frac{1}{u}\frac{du}{\cancel{dx}}\cancel{dx} =$$

$$u = \operatorname{sen} x \qquad \frac{du}{dx} = \cos(x)$$

$$= \int \frac{du}{u} = \ln u =$$

$$= \ln(\operatorname{sen}(x))$$

Studiare $f(x) = \dfrac{x}{1+x^2}$ $\qquad \cancel{\dfrac{x}{1-x^2}}$

Dominio : $\mathbb{R}$

$$\lim_{x\to+\infty} f(x) = 0 \qquad \lim_{x\to-\infty} f(x) = 0$$

$$f'(x) = \frac{1\cdot(1+x^2) - x\cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

punti stazionari $(f'(x)=0)$:

$x = 1 \qquad f(1) = \frac{1}{2}$

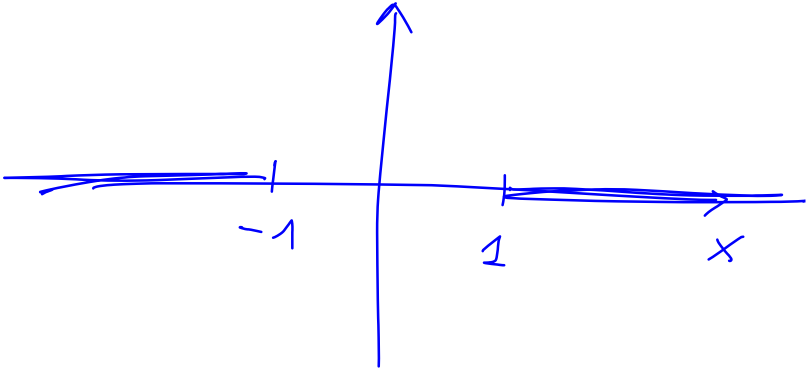
$x = -1 \qquad f(-1) = -\frac{1}{2}$

dove cresce ($f'(x) > 0$) ?

$$f'(x) = \frac{1-x^2}{(1+x^2)^2} > 0 \iff 1-x^2 > 0$$

$$\iff x^2 < 1$$

$$\iff -1 < x < 1$$



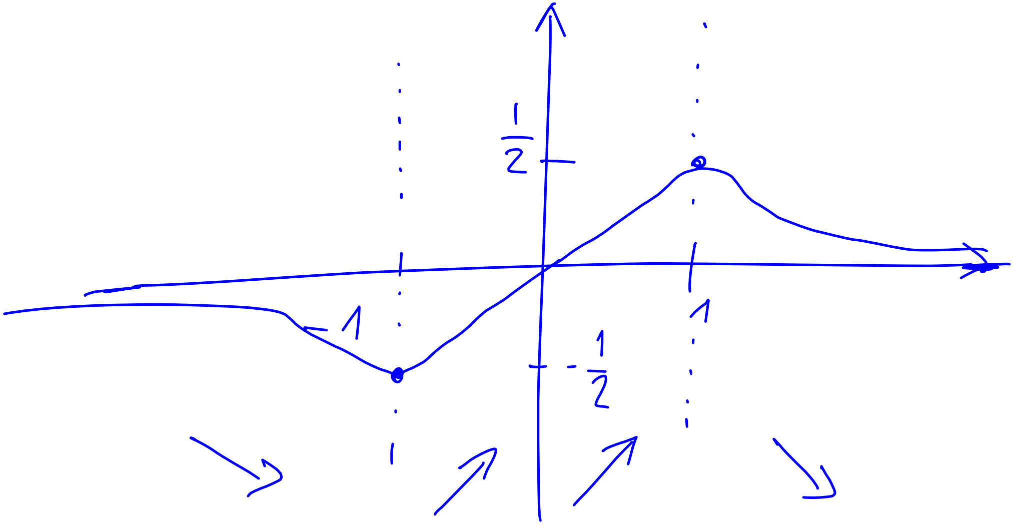
$$1-x^2 > 0 \qquad (1+x)(1-x) > 0 \iff \begin{cases} 1+x>0 \\ 1-x>0 \end{cases}$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$\text{o} \begin{cases} 1+x<0 \\ 1-x<0 \end{cases}$$

$$\begin{cases} 1+x>0 \\ 1-x>0 \end{cases} \qquad \begin{cases} x>-1 \\ x<1 \end{cases} \qquad -1<x<1$$

o $$\begin{cases} 1+x<0 \\ 1-x<0 \end{cases} \qquad \begin{cases} x<-1 \\ x>1 \end{cases}$$ impossibile



$$\lim_{x \to 0} \operatorname{sen}(7x^6) f(x) = \lim_{x \to 0} \left(\frac{\operatorname{sen}(7x^6)}{7x^6} \right) 7x^6 f(x) =$$

$$= \lim_{x \to 0} 7x^6 f(x)$$

$$\lim_{u \to 0} \frac{\operatorname{sen}(u)}{u}$$

$$u = 7x^6 \to 0$$
$$\Longleftrightarrow x \to 0$$

$$\lim_{x \to 0} \ln\left(1 + \underbrace{3x^4}_{}\right) f(x) = \lim_{x \to 0} \frac{\ln(1+3x^4)}{3x^4} \cdot 3x^4 f(x) =$$

$$= \lim_{x \to 0} 3x^4 f(x)$$

$$\lim_{u \to 0} \frac{\ln(1+u)}{u} = 1$$

$$u = 3x^4$$

$$\lim_{x\to 0} \frac{2x^2}{\ln(1+3x^2)} = \lim_{x\to 0} \frac{3x^2}{\underbrace{\ln(1+3x^2)}} \quad \frac{2\cancel{x^2}}{3\cancel{x^2}} = \frac{2}{3}$$

l'Hopital

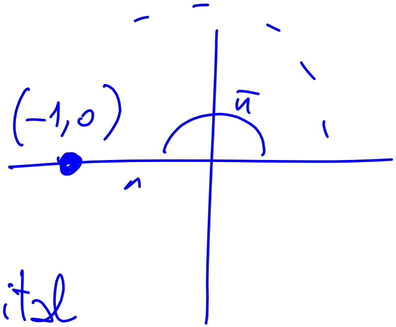
$$= \lim_{x\to 0} \frac{4\cancel{x}}{\frac{1}{1+3x^2} \cdot 6\cancel{x}} = \frac{2}{3}$$

$$\frac{d}{dx} \ln^2(x) = \frac{2}{x} \ln(x) \, .$$

$$\|$$

$$\left(\ln(x)\right)^2$$

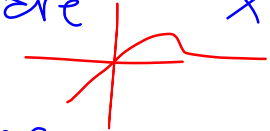
$$\lim_{x\to 0} \frac{\operatorname{sen}(\pi - x^2)}{7x^2 e^x} =$$



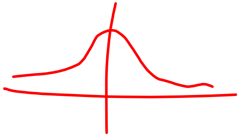
$$= \lim_{x\to 0} \frac{\operatorname{sen}(\pi - x^2)}{7x^2} \overset{\text{l'Hopital}}{=}$$

$$= \lim_{x\to 0} \frac{(-\cancel{2}\cancel{x})\cdot\cos(\pi - \overset{0}{x^2})}{\underset{7}{\cancel{14}}\,\cancel{x}} = \frac{1}{7}$$

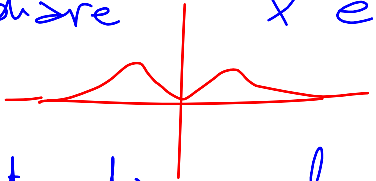
Studiare xe^{-x}



Studiare $\dfrac{1}{1+x^2}$



Studiare $x^2e^{-x^2}$



dominio

$\lim_{x\to\pm\infty}$

derivata

punti stazionari

cresce/decresce

$$\lim_{x\to 1}\frac{\mathrm{sen}(\pi x)}{\ln(x)} = -\pi$$

$$\lim_{x\to 0}\frac{x\,\mathrm{sen}(x)}{(e^{2x}-1)^2} = \frac{1}{4}$$

Derivate di:

$$\ln(\cos(x))$$

$$-\frac{\mathrm{sen}(x)}{\cos(x)} \leftarrow$$

$$\mathrm{sen}(x)\,e^{x}$$

$$\cos(x)e^{x}+\mathrm{sen}(x)e^{x} \leftarrow$$

$$\frac{\cos(x^2)2x^2-\mathrm{sen}(x^2)}{x^2} \leftarrow \frac{\mathrm{sen}(x^2)}{x}$$

$$x\,\mathrm{sen}(e^{x}) \quad \mathrm{sen}(e^{x}) + e^{x}x\cos(e^{x})$$

$$\ln(\ln(x)) = \frac{1}{x\ln(x)}$$

$$\mathrm{sen}(\ln(x))$$

$$\cos(\ln(x))\frac{1}{x}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Trovare $C = A \cdot B = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

$\det C = 3$

$$\frac{1}{3}\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} = C^{-1} \quad , \quad C^2 = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$$

$$C^2 = C \cdot C$$